Methods and their Applications, August 2000, Budapest, Hungary, hep-th/0103007.

b0040 Verlinde E (1988) Fusion rules and modular transformations in 2D conformal field theory. *Nuclear Physics B* 300: 360.

Werner W *Random Planar Curves and Schramm–Loewner Evolutions*, Springer Lecture Notes (to appear), math.PR/0303354. b0045

a0005

Boundary Control Method and Inverse Problems of Wave Propagation

M I Belishev, Petersburg Department of Steklov Institute of Mathematics, St. Petersburg, Russia

s0005 Introduction

p0005 Inverse problems are generally positioned as the problems of determination of a system (its structure, parameters, etc.) from its "input → output" correspondence.

p0010 The boundary-value inverse problems deal with systems which describe processes (wave, heat, electromagnetic ones, etc.) occurring in media occupying a spatial domain. The process is initiated by a boundary source (input) and is described by a solution of a certain partial differential equation in the domain. Certain additional information about the solution, which can be extracted from measurements on the boundary, plays the role of the output. The objective is to determine the parameters of the medium – in particular, the coefficients in the equation – from this information.

p0015 The boundary control (BC) method (Belishev 1986) is an approach to the boundary-value inverse problems based on their links with the control theory and system theory. The present article is a version of the BC method which solves the problem of reconstruction of a Riemannian manifold from its boundary spectral or dynamical data.

s0010 Forward Problems

s0015 Manifold

p0020 Let (Ω, d) be a smooth compact Riemannian manifold with the boundary Γ, $\dim \Omega \geq 2$; d is the distance determined by the metric tensor g. For $A \subset \Omega$ denote

$$\langle A \rangle^r := \{x \in \Omega \,|\, d(x, A) \leq r\}, \quad r \geq 0$$

the hypersurfaces $\Gamma^T := \{x \in \Omega \,|\, d(x, \Gamma) = T\}$, $T > 0$ are equidistant to Γ. In terms of the dynamics of the system, the value

$$T_* := \min\{T > 0 \,|\, \langle \Gamma \rangle^T = \Omega\} = \max_{\Omega} d(\cdot\,, \Gamma)$$

means the time needed for waves, moving from Γ with the unit speed, to fill Ω.

A point $x \in \Omega$ is said to belong to the set $c_0 \subset \Omega$ if x is connected with Γ via more than one shortest geodesic. The set $c := \bar{c}_0$ is called the separation set (cut locus) of Ω with respect to Γ. It is a closed set of zero volume. Let $\tau_*(\gamma)$ be the length of the geodesic emanating from $\gamma \in \Gamma$ orthogonally to Γ and connecting γ with c. The function $\tau_*(\cdot)$ is continuous on Γ. p0025

For $x \in \Omega \setminus c$ the pair (γ, τ), such that $\tau = d(x, \Gamma) = d(x, \gamma)$, constitutes the semigeodesic coordinates of x. The set of these coordinates p0030

$$\Theta := \{(\gamma, \tau) \,|\, \gamma \in \Gamma, 0 \leq \tau < \tau_*(\gamma)\} \subset \Gamma \times [0, T_*]$$

is called the pattern of Ω. Pictorially, to get the pattern, one needs to slit Ω along c and then pull it on the cylinder $\Gamma \times [0, T_*]$. The part $\Theta^T := \Theta \cap (\Gamma \times [0, T])$ of the pattern consists of the semigeodesic coordinates of the points $x \in \langle \Gamma \rangle^T \setminus c$ (**Figure 1**).

Dynamical System s0020

Propagation of waves in the manifold is described by a dynamical system α^T of the form p0035

$$u_{tt} - \Delta_g u = h \qquad \text{in } \Omega \times (0, T) \qquad [1]$$

$$u\,|_{t=0} = u_t\,|_{t=0} = 0 \qquad \text{in } \Omega \qquad [2]$$

$$u = f \qquad \text{on } \Gamma \times [0, T] \qquad [3]$$

where Δ_g is the Beltrami–Laplace operator, $0 < T \leq \infty$, f and h are the boundary and volume sources (controls), $u = u^{f,h}(x,t)$ is the solution (wave).

Set $\mathcal{H} := L_2(\Omega)$; the spaces of the controls are p0040

$$\mathcal{F}^T := L_2(\Gamma \times [0, T]), \quad \mathcal{G}^T := L_2([0, T]; \mathcal{H})$$

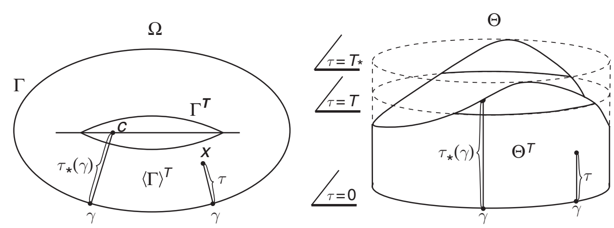

Figure 1 Manifold and pattern. (Data from Belishev (1997).) f0005

The "input ↦ state" map of the system α^T is realized by the control operator W^T:

$$\mathcal{F}^T \times \mathcal{G}^T \to \mathcal{H}, \quad W^T\{f,h\} := u^{f,h}(\cdot, T)$$

and its parts

$$W^T_{\mathrm{bd}} : \mathcal{F}^T \to \mathcal{H}, \quad W^T_{\mathrm{vol}} : \mathcal{G}^T \to \mathcal{H}$$
$$W^T_{\mathrm{bd}} f := u^{f,0}(\cdot, T), W^T_{\mathrm{vol}} h := u^{0,h}(\cdot, T)$$

p0045 In the case $f = 0$ the evolution of the system is governed by the operator $L := -\Delta_g$ defined on the Sobolev class $H^2(\Omega) \cap H^1_0(\Omega)$ of functions vanishing on Γ, and the semigroup representation

$$u^{0,h}(\cdot, r) = W^r_{\mathrm{vol}} h = \int_0^r L^{-1/2} \sin\left[(r-t)L^{1/2}\right] h(\cdot, t)\mathrm{d}t \quad [4]$$

holds for all $r \geq 0$.

p0050 The "input ↦ output" map is implemented by the response operator $R^T : \mathcal{F}^T \to \mathcal{F}^T$,

$$R^T f := \partial_\nu u^{f,0} \quad \text{on } \Gamma \times [0, T]$$

defined on controls $f \in H^1(\Gamma \times [0, T])$ vanishing on $\Gamma \times \{t = 0\}$; here $\nu = \nu(\gamma)$ is the outward normal to Γ. The normal derivative $\partial_\nu u^{f,0}$ describes the forces appearing on Γ as a result of interaction of the wave with the boundary.

p0055 The map $C^T : \mathcal{F}^T \to \mathcal{F}^T, C^T := (W^T_{\mathrm{bd}})^* W^T_{\mathrm{bd}}$, which is called the connecting operator, can be represented via the response operator of the system α^{2T}:

$$C^T = \tfrac{1}{2}(S^T)^* R^{2T} J^{2T} S^T \quad [5]$$

$S^T : \mathcal{F}^T \to \mathcal{F}^{2T}$ being the extension of controls from $\Gamma \times [0, T]$ onto $\Gamma \times [0, 2T]$ as odd functions of t with respect to $t = T$, and $J^{2T} : \mathcal{F}^{2T} \to \mathcal{F}^{2T}$ being the integration

$$(J^{2T} f)(\cdot, t) = \int_0^t f(\cdot, s)\mathrm{d}s$$

s0025 **Controllability**

p0060 Open subsets $\sigma \subset \Gamma$ and $\omega \subset \Omega$ determine the subspaces

$$\mathcal{F}^T_\sigma := \{f \in \mathcal{F}^T \mid \operatorname{supp} f \subset \overline{\sigma} \times [0, T]\}$$
$$\mathcal{G}^T_\omega := \{h \in \mathcal{G}^T \mid \operatorname{supp} h \subset \overline{\omega} \times [0, T]\}$$

of controls acting from σ and ω, respectively. In view of hyperbolicity of the problem [1]–[3], the relation

$$\operatorname{supp} u^{f,h}(\cdot, t) \subset \langle \overline{\sigma} \rangle^t \cup \langle \overline{\omega} \rangle^t, \quad t \geq 0 \quad [6]$$

holds for $f \in \mathcal{F}^T_\sigma$ and $h \in \mathcal{G}^T_\omega$. This means that the waves propagate in Ω with the speed ≤ 1.

The sets of waves p0065

$$\mathcal{U}^T_\sigma := W^T_{\mathrm{bd}} \mathcal{F}^T_\sigma, \qquad \mathcal{U}^T_\omega := W^T_{\mathrm{vol}} \mathcal{G}^T_\omega$$

are said to be reachable at time $t = T$ from σ and ω, respectively. Denoting

$$\mathcal{H}A := \{y \in \mathcal{H} \mid \operatorname{supp} y \subset \overline{A}\}$$

by virtue of [6] one has the embeddings $\mathcal{U}^T_\sigma \subset \mathcal{H}\langle \overline{\sigma} \rangle^T$ and $\mathcal{U}^T_\omega \subset \mathcal{H}\langle \overline{\omega} \rangle^T$. The property of the system α^T that plays the key role in inverse problems is that these embeddings are dense:

$$\mathrm{cl}\,\mathcal{U}^T_\sigma = \mathcal{H}\langle \overline{\sigma} \rangle^T, \qquad \mathrm{cl}\,\mathcal{U}^T_\omega = \mathcal{H}\langle \overline{\omega} \rangle^T \quad [7]$$

for any $T > 0$ (cl denotes the closure in $\mathcal{H}$).

In control theory, relations [7] are interpreted as an approximate controllability of the system in subdomains filled with waves; the name "BC method" is derived from the first one (boundary controllability). This property means that the sets of waves are rich enough: any function supported in the subdomain $\langle \overline{\sigma} \rangle^T$ reachable for waves excited on σ can be approximated with any precision in $\mathcal{H}$-norm by the wave $u^{f,0}(\cdot, T)$ due to appropriate choice of the control f acting from σ. The proof of [7] relies on the fundamental Holmgren–John–Tataru unique continuation theorem for the wave equation (Tataru 1993). p0070

Laplacian on Waves s0030

If $h = 0$, so that the system is governed only by boundary controls, its trajectory $\{u^{f,0}(\cdot, t) | 0 \leq t \leq T\}$ does not leave the reachable set $\mathcal{U}^T_\Gamma$. In this case, the system possesses one more intrinsic operator L^T which acts in the subspace $\mathrm{cl}\,\mathcal{U}^T_\Gamma$ and is introduced through its graph p0075

$$\mathrm{gr}\, L^T := \mathrm{cl}\left\{\{W^T_{\mathrm{bd}} f, -W^T_{\mathrm{bd}} f_{tt}\} | f \in C^\infty_0(\Gamma \times (0, T))\right\} \quad [8]$$

(closure in $\mathcal{H} \times \mathcal{H}$). By virtue of the relation $L^T W^T_{\mathrm{bd}} f = -\Delta_g W^T_{\mathrm{bd}} f$ following from the wave equation [1] and [6], the operator L^T is interpreted as Laplacian on waves filling the subdomain $\langle \Gamma \rangle^T$.

In the case $T > T_*$, one has $\langle \Gamma \rangle^T = \Omega$, $\mathrm{cl}\,\mathcal{U}^T_\Gamma = \mathcal{H}$, and L^T is a densely defined operator in $\mathcal{H}$, satisfying $L^T \subset L$. Using [7], one proves the equality $L^T = L$. This equality and representation [4] imply that p0080

$$W^r_{\mathrm{vol}} h = \int_0^r (L^T)^{-1/2} \sin\left[(r-t)(L^T)^{1/2}\right] h(\cdot, t)\mathrm{d}t \quad [9]$$

for all $r \geq 0$ and any fixed $T > T^*$.

s0035 ### Spectral Problem

p0085 The Dirichlet homogeneous boundary-value problem is to find nontrivial solutions of the system

$$-\Delta_g \varphi = \lambda\varphi \quad \text{in } \Omega \qquad [10]$$

$$\varphi = 0 \quad \text{on } \Gamma \qquad [11]$$

This problem is equivalent to the spectral analysis of the operator L; it has the discrete spectrum $\{\lambda_k\}_{k=1}^{\infty}, 0 < \lambda_1 < \lambda_2 \leq \cdots, \lambda_k \to \infty$; the eigenfunctions $\{\varphi_k\}_{k=1}^{\infty}, L\varphi_k = \lambda_k\varphi_k$, form an orthonormal basis in $\mathcal{H}$.

p0090 Expanding the solutions of the problem (1)–(3) over the eigenfunctions of the problem [10], [11] one derives the spectral representation of waves:

$$u^{f,0}(\cdot, T) = W_{\text{bd}}^T f = \sum_{k=1}^{\infty} (f, s_k^T)_{\mathcal{F}^T} \varphi_k(\cdot) \qquad [12]$$

where

$$s_k^T(\gamma, t) := \lambda_k^{-1/2} \sin\left[(T - t)\lambda_k^{1/2}\right] \partial_\nu \varphi_k(\gamma)$$

Thus, for a given control f, the Fourier coefficients of the wave $u^{f,0}$ are determined by the spectrum $\{\lambda_k\}_{k=1}^{\infty}$ and the derivatives $\{\partial_\nu \varphi_k\}_{k=1}^{\infty}$.

s0040 ## Inverse problems

s0045 ### General Setup

p0095 The set of pairs $\Sigma := \{\lambda_k; \partial_\nu \varphi_k\}_{k=1}^{\infty}$ associated with the problem [10], [11] is said to be the Dirichlet spectral data of the manifold (Ω, d). The spectral (frequency domain) inverse problem is to recover the manifold from its spectral data.

p0100 Since the speed of wave propagation is unity, the response operator R^T contains the information not about the entire manifold but only about its part $\langle \Gamma \rangle^{T/2}$. This fact is taken into account in the dynamical (time domain) inverse problem which aims to recover the manifold from the operator R^{2T} given for a fixed $T > T_*$.

p0105 If the manifolds (Ω', d') and (Ω'', d'') are isometric via an isometry $i : \Omega' \to \Omega''$, then, identifying the boundaries by $i(\gamma) \equiv \gamma$, one gets two manifolds with the common boundary $\Gamma = \partial\Omega' = \partial\Omega''$ which possess identical inverse data: $\Sigma' = \Sigma'', R'^{2T} = R''^{2T}$. Such manifolds are called equivalent: they are indistinguishable for the external observer extracting Σ or R^{2T} from the boundary measurements. Therefore, these data do not determine the manifold uniquely and both of the inverse problems need to be clarified. The precise formulation is given in the form of two questions:

1. Does the coincidence of the inverse data imply the equivalence of the manifolds?
2. Given the inverse data of an unknown manifold, how to construct a manifold possessing these data?

The BC method gives an affirmative answer to the first question and provides a procedure producing a representative of the class of equivalent manifolds from its inverse data. The method is based on the concepts of model and "coordinatization."

Model s0050

A pair consisting of an auxiliary Hilbert space $\tilde{\mathcal{H}}$ and an operator $\tilde{W}_{\text{bd}}^T : \mathcal{F}^T \to \tilde{\mathcal{H}}$ is said to be a model of the system α^T, if $\tilde{W}_{\text{bd}}^T$ is determined by inverse data, and the map $U : W_{\text{bd}}^T f \mapsto \tilde{W}_{\text{bd}}^T f$ is an isometry from Ran $W_{\text{bd}}^T \subset \mathcal{H}$ onto Ran $\tilde{W}_{\text{bd}}^T \subset \tilde{\mathcal{H}}$. The model is an intermediate object in solving inverse problems. It plays the role of an auxiliary copy of the original dynamical system which an external observer can build from measurements on the boundary. While the genuine wave process inside Ω, initiated by a boundary control, remains unaccessible for direct measurements, its $\tilde{\mathcal{H}}$-representation can be visualized by means of the model control operator $\tilde{W}_{\text{bd}}^T$. This is illustrated by the diagram on **Figure 2**, where the upper part is invisible for an external observer, whereas the lower part can be extracted from inverse data. p0110

Each type of data determines a corresponding model. The spectral model is the pair p0115

$$\tilde{\mathcal{H}} := l_2, \qquad \tilde{W}_{\text{bd}}^T := \{(\cdot, s_k^T)_{\mathcal{F}^T}\}_{k=1}^{\infty} \qquad [13]$$

(see [12]); the role of isometry U is played by the Fourier transform $F : \mathcal{H} \to \tilde{\mathcal{H}}, Fy := \{(y, \varphi)_{\mathcal{H}}\}_{k=1}^{\infty}$. By virtue of [4], the data Σ also determine the operator $\tilde{W}^r{}_{\text{vol}} : L_2([0, r]; \tilde{\mathcal{H}}) \to \tilde{\mathcal{H}}$,

$$\tilde{W}^r{}_{\text{vol}} = \int_0^r \tilde{L}^{-1/2} \sin\left[(r - t)(\tilde{L})^{1/2}\right](\cdot)(t)\, dt, \quad r \geq 0 \qquad [14]$$

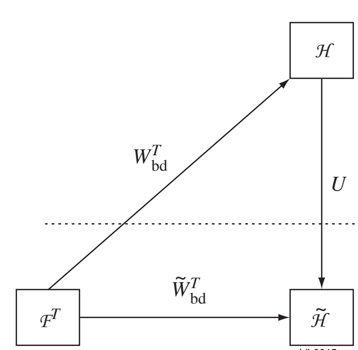

Figure 2 Model of a system. (Data from Belishev (1997).) f0010

where $\tilde{L} := ULU^* = \mathrm{diag}\{\lambda_k\}_{k=1}^{\infty}$. Thus, the spectral model allows one to see the Fourier images of invisible waves.

p0120 According to [5], the response operator R^{2T} determines the modulus of the control operator

$$|W^T_{\mathrm{bd}}| = [(W^T_{\mathrm{bd}})^* W^T_{\mathrm{bd}}]^{1/2} = (C^T)^{1/2}$$

which enters in the polar decomposition $W^T_{\mathrm{bd}} = \Phi |W^T_{\mathrm{bd}}|$. Along with it, the response operator determines the dynamical model

$$\tilde{\mathcal{H}} := \mathrm{cl}\ \mathrm{Ran}(C^T)^{1/2}, \qquad \tilde{W}^T_{\mathrm{bd}} := (C^T)^{1/2} \qquad [15]$$

The correspondence "system → model" is realized by the isometry $U = \Phi^* : W^T_{\mathrm{bd}} f \mapsto |W^T_{\mathrm{bd}}| f$. The operator $\tilde{L}^T := UL^T U^*$ dual to the Laplacian on waves, is determined by its graph

$$\mathrm{gr}\,\tilde{L}^T := \mathrm{cl}\left\{ \{\tilde{W}^T_{\mathrm{bd}} f, -\tilde{W}^T_{\mathrm{bd}} f_{tt}\} \,|\, f \in C_0^\infty(\Gamma \times (0,T)) \right\} \qquad [16]$$

(see [8]) and, therefore, $\tilde{L}^T$ is also determined by R^{2T}. In the case $T > T_*$, the operator $\tilde{W}^r_{\mathrm{vol}} : L_2([0,r]; \tilde{\mathcal{H}}) \to \tilde{\mathcal{H}}$ dual to $\tilde{W}^r_{\mathrm{vol}}$, is represented in the form

$$\tilde{W}^r_{\mathrm{vol}} = \int_0^r (\tilde{L}^T)^{-1/2} \sin\left[(r-t)(\tilde{L}^T)^{1/2}\right](\cdot)(t)\,\mathrm{d}t, \quad r \geq 0 \qquad [17]$$

in accordance with [9]. Thus, the dynamical model visualizes the Φ^*-images of the waves propagating inside Ω.

s0055 **Wave Coordinatization**

p0125 In a general sense, a coordinatization is a correspondence between points x of the studied set $\mathcal{A}$ and elements $\tilde{x}$ of another set $\tilde{\mathcal{A}}$ such that: (i) the elements of $\tilde{\mathcal{A}}$ are accessible and distinguishable; (ii) the map $x \mapsto \tilde{x}$ is a bijection; and (iii) relations between elements of $\tilde{\mathcal{A}}$ determine those between points of $\mathcal{A}$ which are studied (H Weyl). Coordinatization enables one to study $\mathcal{A}$ via operations with coordinates $\tilde{x} \in \tilde{\mathcal{A}}$.

p0130 The external observer investigating the manifold probes Ω with waves initiated by sources on Γ. The relevant coordinatization of Ω described below uses such waves and is implemented in three steps.

p0135 *Step 1* (*subdomains*) Let $x(\gamma, \tau)$ be the end point of the geodesic of the length $\tau > 0$ emanating from $\gamma \in \Gamma$ in the direction $-\nu(\gamma)$, and let $\sigma^\varepsilon_\gamma \subset \Gamma$ be a small neighborhood shrinking to γ as $\varepsilon \to 0$. If $\tau \leq \tau_*(\gamma)$, then the family of subdomains

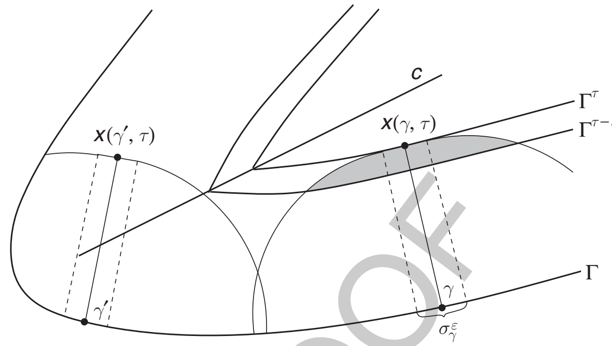

Figure 3 The subdomains. f0015

$$\omega^\varepsilon(\gamma, \tau) := [\langle\Gamma\rangle^\tau \setminus \langle\Gamma\rangle^{\tau-\varepsilon}] \cap \langle\sigma^\varepsilon_\gamma\rangle^\tau$$

(shaded domain on **Figure 3**) shrinks to $x(\gamma, \tau)$; if $\tau > \tau_*(\gamma)$, then the family terminates: $\omega^\varepsilon(\gamma, \tau) = \emptyset$ as $\varepsilon < \varepsilon_0(\gamma)$ (the case $\gamma = \gamma'$ in **Figure 3**). Such behavior of subdomains implies that

$$\lim_{\varepsilon \to 0} \left\langle [\langle\Gamma\rangle^\tau \setminus \langle\Gamma\rangle^{\tau-\varepsilon}] \cap \langle\sigma^\varepsilon_\gamma\rangle^\tau \right\rangle^r = \begin{cases} \langle x(\gamma,\tau)\rangle^r, & \tau \leq \tau_*(\gamma) \\ \emptyset, & \tau > \tau_*(\gamma) \end{cases} \qquad [18]$$

Step 2 (*wave subspaces*) Pass from the subdomains to the corresponding subspaces $\mathcal{H}\langle\Gamma\rangle^\tau, \mathcal{H}\langle\sigma^\varepsilon_\gamma\rangle^\tau$, $\mathcal{H}\langle\omega^\varepsilon(\gamma,\tau)\rangle^r$, and represent them via reachable sets by [7]:

$$\mathcal{H}\langle\Gamma\rangle^\tau = \mathrm{cl}\ W^\tau_{\mathrm{bd}}\mathcal{F}^\tau, \qquad \mathcal{H}\langle\sigma^\varepsilon_\gamma\rangle^\tau = \mathrm{cl}\ W^\tau_{\mathrm{bd}}\mathcal{F}^\tau_{\sigma^\varepsilon_\gamma}$$

$$\begin{aligned} \mathcal{H}\langle\omega^\varepsilon(\gamma,\tau)\rangle^r &= \mathrm{cl}\ W^r_{\mathrm{vol}} L_2([0,r]; \mathcal{H}\omega^\varepsilon(\gamma,\tau)) \\ &= \mathrm{cl}\ W^r_{\mathrm{vol}} L_2\Big([0,r]; [\mathcal{H}\langle\Gamma\rangle^\tau \ominus \mathcal{H}\langle\Gamma\rangle^{\tau-\varepsilon}] \cap \mathcal{H}\langle\sigma^\varepsilon_\gamma\rangle^\tau\Big) \\ &= \mathrm{cl}\ W^r_{\mathrm{vol}} L_2\Big([0,r]; [\mathrm{cl}\ W^\tau_{\mathrm{bd}}\mathcal{F}^\tau \ominus \mathrm{cl}\ W^{\tau-\varepsilon}_{\mathrm{bd}}\mathcal{F}^{\tau-\varepsilon}] \cap \mathrm{cl}\ W^\tau_{\mathrm{bd}}\mathcal{F}^\tau_{\sigma^\varepsilon_\gamma}\Big) \end{aligned}$$

Define

$$\mathcal{W}^r_{(\gamma,\tau)} := \lim_{\varepsilon\to 0} \mathrm{cl}\ W^r_{\mathrm{vol}} L_2\Big([0,r]; [\mathrm{cl}\ W^\tau_{\mathrm{bd}}\mathcal{F}^\tau \ominus\ \mathrm{cl}\ W^{\tau-\varepsilon}_{\mathrm{bd}}\mathcal{F}^{\tau-\varepsilon}] \cap \mathrm{cl}\ W^\tau_{\mathrm{bd}}\mathcal{F}^\tau_{\sigma^\varepsilon_\gamma}\Big) \qquad [19]$$

$\mathcal{W}^r_{(\gamma,0)} := \mathcal{W}^r_{(\gamma,+0)}$, $r \geq 0$ (the limits in the sense of the strong operator convergence of the projections in $\mathcal{H}$ on the corresponding subspaces). By the definitions, one has $\mathcal{W}^r_{(\gamma,\tau)} = \lim_{\varepsilon\to 0} \mathcal{H}\langle\omega^\varepsilon(\gamma,\tau)\rangle^r$, whereas [18] leads to the equality

$$\mathcal{W}^r_{(\gamma,\tau)} = \begin{cases} \mathcal{H}\langle x(\gamma,\tau)\rangle^r, & \tau \leq \tau_*(\gamma) \\ \{0\}, & \tau > \tau_*(\gamma) \end{cases} \qquad [20]$$

for all $\gamma \in \Gamma$, $\tau \geq 0$, $r \geq 0$. As a result, since any $x \in \Omega$ can be represented as $x = x(\gamma, \tau)$, one attaches to every point of the manifold a family of expanding subspaces $\{\mathcal{W}^r_{(\gamma,\tau)} | r \geq 0\}$ built out of waves. As is seen from [20], the family is determined by the point x (not dependent on the representation $x = x(\gamma,\tau)$); the subspaces which it consists of coincide with $\mathcal{H}\langle x \rangle^r$.

p0140 Expressing the distance as

$$d(x', x'') = 2 \inf \{r > 0 \,|\, \mathcal{H}\langle x' \rangle^r \cap \mathcal{H}\langle x'' \rangle^r \neq \{0\}\}$$

in accordance with [20], one can represent

$$d(x', x'') = 2 \inf \{r > 0 \,|\, \mathcal{W}^r_{(\gamma',\tau')} \cap \mathcal{W}^r_{(\gamma'',\tau'')} \neq \{0\}\} \qquad [21]$$

where $x' = x(\gamma', \tau')$, $x'' = x(\gamma'', \tau'')$, and hence find the distance via the above families.

Step 3 (*wave copy*) By varying $\gamma \in \Gamma, \tau \geq 0$, gather all nonzero families $\{\mathcal{W}^r_{(\gamma,\tau)} | r \geq 0\} =: \tilde{x}$ in the set $\tilde{\Omega} = \{\tilde{x}\}$. Redenoting $\mathcal{W}^r_{\tilde{x}} := \mathcal{W}^r_{(\gamma,\tau)} \in \tilde{x}$, endow the set with the distance

$$\tilde{d}(\tilde{x}', \tilde{x}'') := 2 \inf \{r > 0 \,|\, \mathcal{W}^r_{\tilde{x}'} \cap \mathcal{W}^r_{\tilde{x}''} \neq \{0\}\} \qquad [22]$$

In view of [21], one has $d(x', x'') = \tilde{d}(\tilde{x}', \tilde{x}'')$, so that the metric space $(\tilde{\Omega}, \tilde{d})$ is an isometric copy of (Ω, d) by construction. Thus, the correspondence $x \mapsto \tilde{x}$ ("point $\mapsto$ family") is an isometry and satisfies the general principles (i)–(iii) of coordinatization.

p0145 The manifold $(\tilde{\Omega}, \tilde{d})$ is the end product of the wave coordinatization. It represents the original manifold as a collection of infinitesimally small sources interacting with each other via the waves which they produce.

s0060 Solving Inverse Problems

p0150 The motivation for the above coordinatization is that the wave copy can be reproduced via any model. Namely, the external observer with the knowledge of Σ or $R^{2T} (T > T_*)$ can recover $(\tilde{\Omega}, \tilde{d})$ up to isometry by the following procedure:

1. Construct the model corresponding to the given inverse data and determine the operators $\tilde{W}^\tau_{\text{bd}}$, $0 \leq \tau \leq T$ by [13], [15]; then determine $\tilde{L}, \tilde{L}^T$, and $\tilde{W}^r_{\text{vol}}$ by [14] or [16], [17].
2. Replace on the right-hand side of [19] all operators W without tildes by the ones with tildes, and get the subspaces $\tilde{\mathcal{W}}^r_{(\gamma,\tau)} = U\mathcal{W}^r_{(\gamma,\tau)}$, $\gamma \in \Gamma$, $\tau \geq 0$, $r \geq 0$.
3. Gather all nonzero families $\{\tilde{\mathcal{W}}^r_{(\gamma,\tau)} | r \geq 0\} =: \hat{x}$ in the set $\hat{\Omega} = \{\hat{x}\}$ and redenote the subspaces as $\hat{\mathcal{W}}^r_{\hat{x}} := \tilde{\mathcal{W}}^r_{(\gamma,\tau)} \in \hat{x}$; endow the set with the metric $\hat{d}(\hat{x}', \hat{x}'') := \inf\{r > 0 \,|\, \tilde{\mathcal{W}}^r_{\hat{x}'} \cap \tilde{\mathcal{W}}^r_{\hat{x}''} \neq \{0\}\}$ (see [22]), and get a sample $(\hat{\Omega}, \hat{d})$ of the wave copy $(\tilde{\Omega}, \tilde{d})$.

This sample is isometric to the original (Ω, d) by construction. Identifying properly the boundaries $\partial\hat{\Omega}$ and Γ, one turns $(\hat{\Omega}, \hat{d})$ into a canonical representative of the class of equivalent manifolds possessing the given inverse data. p0155

If the response operator R^{2T} is given for a fixed $T < T_*$, the above procedure produces the wave copy of the submanifold $(\langle \Gamma \rangle^T, d)$. This locality in time is an intrinsic feature and advantage of the BC method: longer time of observation on Γ increases the depth of penetration into Ω. p0160

Amplitude Formula s0065

Another variant of the BC method is based on geometrical optics formulas describing the propagation of singularities of the waves. p0165

Let $y \in \mathcal{H}$, and let β be the density of the volume in semigeodesic coordinates: $dx = \beta \, d\Gamma \, d\tau$; the function p0170

$$\tilde{y}(\gamma, \tau) := \begin{cases} \beta^{1/2}(\gamma, \tau)\, y(x(\gamma, \tau)), & (\gamma, \tau) \in \Theta \\ 0, & \text{otherwise} \end{cases}$$

defined on $\Gamma \times [0, T_*]$ is called the image of y. The amplitude formula represents the images of waves initiated by boundary controls in the form

$$\widetilde{u^{f,0}(\cdot, T)}(\gamma, \tau) = \lim_{t \to T - \tau - 0} [(W^T_{\text{bd}})^* (I - P^\tau) W^T_{\text{bd}} f](\gamma, t)$$
$$0 < \tau < T$$

where I is the identity operator and P^τ is the projection in $\mathcal{H}$ onto $\text{cl}\, W^T_{\text{bd}} \mathcal{F}^T$. The formula is derived by the ray method going back to J Hadamard, the derivation uses the controllability [7].

Any model determines the right-hand side of the last relation by the isometry: $(W^T_{\text{bd}})^* (I - P^\tau) W^T_{\text{bd}} = (\tilde{W}^T_{\text{bd}})^* (\tilde{I} - \tilde{P}^\tau) \tilde{W}^T_{\text{bd}}$, where $\tilde{W}^T_{\text{bd}} = U W^T_{\text{bd}}$, $\tilde{I}$ is the identity operator, and $\tilde{P}^\tau = U P^\tau U^*$ is the projection in $\tilde{\mathcal{H}}$ onto $\text{cl}\, \tilde{W}^T_{\text{bd}} \mathcal{F}^T$. This leads to the representation p0175

$$\widetilde{u^{f,0}(\cdot, T)}(\gamma, \tau) = \lim_{t \to T - \tau - 0} [(\tilde{W}^T_{\text{bd}})^* (\tilde{I} - \tilde{P}^\tau) \tilde{W}^T_{\text{bd}} f](\gamma, t)$$
$$0 < \tau < T \qquad [23]$$

and makes the amplitude formula a useful tool for solving the inverse problems. The external observer can construct a model via inverse data and then visualize by [23] the wave images on the part Θ^T of the pattern (see **Figure 1**). The collection of images $\widetilde{u^{f,0}}$ corresponding to all possible controls f is rich enough for recovering the tensor g on Θ^T (i.e., the metric tensor in semigeodesic coordinates) and turning the pattern into an isometric copy of the submanifold $(\langle \Gamma \rangle^T, d)$. This variant of the method is more appropriate if one needs to recover unknown coefficients of the wave equation in Ω – it can be realized in terms of numerical algorithms.

s0070 Extensions of the Method

p0180 Electromagnetic waves are also well suited for coordinatization and for constructing the wave copy $(\tilde{\Omega}, \tilde{d})$. An appropriate version of the amplitude formula also exists for the system governed by the Maxwell equations (see Further Reading). At present (2004), the applicability of the BC method to three-dimensional inverse problems of elasticity theory is still an open question. The following hypothesis concerns the Lamé system: the wave coordinatization procedure (steps 1–3) using the elastic waves instead of the above $u^{f,0}$, gives rise to the copy of $\Omega \subset \mathbf{R}^3$ endowed with the metric $|dx|^2/c_p^2$ where $c_p = \sqrt{(\lambda + 2\mu)/\rho}$ is the speed of the pressure waves.

p0185 The concept of model is used for solving inverse problems for the heat and Schrödinger equations (Avdonin and Belishev, 1995–2004), as well as for the problem of boundary data continuation (Belishev 2001, Kurylev and Lassas 2002). A variant of the BC method allows one to recover not only the manifold but also the Schrödinger type operators on it and/or the dissipative term in the scalar wave equation (Kurylev and Lassas 1993–2003).

p0190 An appropriate version of the amplitude formula solves the inverse problem for one-dimensional two-velocity dynamical system which describes the waves consisting of two modes propagating with different speeds and interacting with each other (Belishev, Blagoveschenskii, Ivanov, 1997–2000).

p0195 One more variant of coordinatization going back to the first paper on the BC method, associates with points $x \in \Omega$ the Dirac measures δ_x; then, their images $\tilde{\delta}_x$ are identified via suitable models. This variant solves inverse problems on graphs and the two-dimensional elliptic Calderon problem. The reader is referred to articles by the present author listed in Further Reading.

p0200 Within the scope of the method, one derives some natural analogs of the classical Gelfand–Levitan–Krein–Marchenko equations (Belishev, 1987–2001). Also, an appropriate analog solves the kinematic inverse problem for a class of two-dimensional manifolds (Pestov 2004).

There exists an abstract version of the approach, embedding the BC method into the framework of linear system theory (Belishev 2001). The method is also related to the problem of triangular factorization of operators (Belishev and Pushnitski 1996). p0205

Numerical algorithms for solving two-dimensional spectral and dynamical inverse problems for the wave equation $\rho u_{tt} - \Delta u = 0$ which recover the variable density ρ have been developed and tested (Filippov, Gotlib, Ivanov, 1994–1999). p0210

Further Reading

Belishev MI (1988) On an approach to multidimensional inverse problems for the wave equation. *Soviet Mathematics. Doklady* 36(3): 481–484. b0005

Belishev MI (1996) Canonical model of a dynamical system with boundary control in the inverse problem of heat conductivity. *St. Petersburg Mathematical Journal* 7(6): 869–890. b0010

Belishev MI (1997) Boundary control in reconstruction of manifolds and metrics. *Inverse Problems* 13(5): R1–R45. b0015

Belishev MI (2001) Dynamical systems with boundary control: models and characterization of inverse data. *Inverse Problems* 17: 659–682. b0020

Belishev MI (2002) How to see waves under the Earth surface (the BC-method for geophysicists). In: Kabanikhin SI and Romanov VG (eds.) *Ill-Posed and Inverse Problems*, pp. 67–84. Utrecht/Boston: VSP. b0025

Belishev MI (2003) The Calderon problem for two-dimensional manifolds by the BC-method. *SIAM Journal of Mathematical Analysis* 35(1): 172–182. b0030

Belishev MI (2004) Boundary spectral inverse problem on a class of graphs (trees) by the BC-method. *Inverse Problems* 20(3): 647–672. b0035

Belishev MI and Glasman AK (2001) Dynamical inverse problem for the Maxwell system: recovering the velocity in the regular zone (the BC-method). *St. Petersburg Mathematical Journal* 12(2): 279–319. b0040

Belishev MI and Gotlib VYu (1999) Dynamical variant of the BC-method: theory and numerical testing. *Journal of Inverse and Ill-Posed Problems* 7(3): 221–240. b0045

Belishev MI, Isakov VM, Pestov LN, and Sharafutdinov VA (2000) On reconstruction of metrics from external electromagnetic measurements. *Russian Academy of Sciences. Doklady. Mathematics* 61(3): 353–356. b0050

Belishev MI and Ivanov SA (2002) Characterization of data of dynamical inverse problem for two-velocity system. *Journal of Mathematical Sciences* 109(5): 1814–1834. b0055

Belishev MI and Lasiecka I (2002) The dynamical Lamé system: regularity of solutions, boundary controllability and boundary data continuation. *ESAIM COCV* 8: 143–167. b0060

Katchalov A, Kurylev Y, and Lassas M (2001) Inverse Boundary Spectral Problems. *Chapman and Hall/CRC Monographs and Surveys in Pure and Applied Mathematics*, vol. 123. Boca Raton, FL: Chapman and Hall/CRC. b0065