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Boundary spectral inverse problem on a class of graphs (trees) by the BC method

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Abstract
A planar graph consisting of strings of variable densities is considered. The spectrum of the Dirichlet problem on the graph and the values of derivatives of the (normalized) eigenfunctions at the boundary vertices form the spectral data. We show that the graph without cycles (tree) and the densities of its edges are determined by the spectral data uniquely up to a natural isometry in the plane. In the framework of our approach (boundary control method; Belishev 1986) we study the boundary controllability of the dynamical system associated with the graph and governed by the wave equation, and exploit this property for recovering the tree from its spectral data. The approach can be extended to a wide class of inverse problems on trees.

Dedicated to A S Blagovestchenskii on the occasion of his 65th birthday

1. Introduction

1.1. About the paper

The mechanical object inspiring the problem is a planar graph consisting of a finite set of strings of variable densities. The graph can be displaced in the normal (with respect to the plane) direction whereas its boundary vertices are rigidly fixed on the plane. This system can vibrate; it possesses the spectrum of the eigenfrequencies and the corresponding eigenmodes.

Assume that we know the spectrum of the graph and the values of derivatives of the eigenmodes at the boundary vertices. What information about the graph may be extracted from these data? We show that, if the graph does not contain cycles (is a tree), such data determine the graph (its structure, lengths of strings and their densities) uniquely, up to a natural isometry in the plane. Moreover, a procedure recovering the tree is proposed.

As each variant of the BC method, our approach invokes results of the boundary control theory (see [5]). We show that the dynamical system associated with a tree and governed by the wave equation is controllable from the boundary in an appropriate sense, and exploit this property in the recovering procedure.

At the end of the paper we mention a wide class of inverse problems on graphs which may be reduced to the spectral problem. The list of possible applications of these problems includes nondestructive testing of the elastic and electric networks, nanoelectronics and synthesis of networks with prescribed characteristics in engineering.

1.2. Statement and main result

Let $\Omega \subset \mathbf{R}^2$ be a finite connected planar graph with edges $\{e\} = E$ (intervals of straight lines) and boundary vertices $\{\gamma_1, \ldots, \gamma_n\} = \Gamma$. The graph is equipped with density ρ which is a strictly positive function on Ω and C^2-smooth on the edges. **See endnote 1**

With the graph Ω one associates a (real) Hilbert space $\mathcal{H} = L_2(\Omega; \rho\,|\,\mathrm{d}x|)$ ($|\,\mathrm{d}x|^2$ is the length element on Ω induced by the $\mathbf{R}^2$-metric), the Sobolev class $\mathcal{H}_0^1 = \{y \in \mathcal{H} \mid y \in C(\Omega); y|_e \in H^1(e), e \in E; y|_\Gamma = 0\}$ and a positive definite bilinear form $l : \mathcal{H} \times \mathcal{H} \mapsto \mathbf{R}$, Dom $l = \mathcal{H}_0^1 \times \mathcal{H}_0^1$,

$$l[y, v] = \sum_{e \in E} \int_e \frac{\mathrm{d}y}{\mathrm{d}e}\frac{\mathrm{d}v}{\mathrm{d}e}\,|\mathrm{d}x|.$$

Let L be the self-adjoint operator in $\mathcal{H}$ corresponding to the form l; on the edges it acts by the rule

$$(Ly)|_e = -\frac{1}{\rho}\frac{\mathrm{d}^2 y}{\mathrm{d}e^2}.$$

The operator L has the discrete spectrum $\{\lambda_k\}_{k=1}^{\infty} : 0 < \lambda_1 < \lambda_2 \leqslant \cdots$; let $\varphi_1, \varphi_2, \ldots$ be the corresponding eigenfunctions orthonormalized in $\mathcal{H}$. Denote

$$\frac{\mathrm{d}\varphi_k}{\mathrm{d}e}\Big|_\Gamma = \mathrm{col}\left\{\frac{\mathrm{d}\varphi_k}{\mathrm{d}e}(\gamma_1), \ldots, \frac{\mathrm{d}\varphi_k}{\mathrm{d}e}(\gamma_n)\right\};$$

the set of pairs $\left\{\lambda_k; \frac{\mathrm{d}\varphi_k}{\mathrm{d}e}\big|_\Gamma\right\}_{k=1}^{\infty}$ is said to be the (Dirichlet) *spectral data* of the (equipped) graph Ω.

We say that two graphs Ω' and Ω'' equipped with densities ρ' and ρ'' are *spatially isometric* if there exists an isometry I connecting the metric spaces $(\Omega', |\mathrm{d}x|^2)$ and $(\Omega'', |\mathrm{d}x|^2)$ such that $I(\Omega') = \Omega''$ and $\rho'' \circ I = \rho'$. As is easy to see, spatially isometric graphs have one and the same spectral data. The main result of the paper is that in the case of trees the converse turns out to be true.

Theorem 1. *If the spectral data of two trees coincide, the trees are spatially isometric.*

In other words, the tree is determined by its spectral data up to a spatial isometry.

1.3. Remarks

To the best of our knowledge, an investigation of inverse problems on graphs was initially given by Gerasimenko in [12]. The reader can find a review of results in this area in [14]. Most of them concern the case where the graph is given whereas its parameters (densities, potentials) are required to be recovered. The examples of nonuniqueness to the problem of reconstruction of the unknown graph are presented in [14]; all of them concern graphs with cycles. A peculiarity of our paper is that we deal with an *unknown* graph equipped with an *unknown* density and, if the graph is a tree, recover both.

At the end of the paper we discuss some related inverse problems with another kind of data which can be solved by reducing to the spectral problem.

In order to restrict the volume of the paper, we omit some of the technical details (mainly in sections 3.2–3.4). In particular, we do not prove the results given in propositions: they are either well known or their proofs should present no problems.

2. Graph

2.1. Terminology

An *edge* is a finite interval of a straight line on the plane $\mathbf{R}^2$:

$$e = \{\mathrm{col}\{x^1, x^2\} \mid x^i = a^i + b^i s, a^i, b^i = \mathrm{const},\ s \in (0,1)\};$$

a *vertex* is a point $v \in \mathbf{R}^2$. A *graph* $\Omega \subset \mathbf{R}^2$ is a disjoint sum $\Omega = E \cup V$ of a finite set of nonintersecting edges $E = \{e\}$ and a finite set of vertices $V = \{v\}$ such that Ω is a closed connected set on the plane.

We say that $e \in E$ is *incident* to $v \in V$ and write $e \sim v$ if $v \in \bar{e}$ (the closure in $\mathbf{R}^2$) and denote $r(v) := \sharp\{e \in E \mid e \sim v\}$. Just for simplicity we consider only the graphs satisfying $r(\cdot) \neq 2$.

The set $\Gamma := \{v \in V \mid r(v) = 1\}$ is the *boundary* of Ω; the points of the set $\mathrm{int}\,\Omega := \Omega\backslash\Gamma$ are *interior points* of the graph.

A *path* $\pi[a,b] \subset \Omega$ connecting points $a, b \in \Omega$ is the image of the segment [0,1] through a homeomorphism $\kappa : [0,1] \mapsto \Omega$ such that $\kappa(0) = a, \kappa(1) = b$; the notation $\pi[a,b), \pi(a,b], \pi(a,b)$ is also of the clear meaning.

A graph Ω is a *tree* if for any points $a, b \in \Omega, a \neq b$ there exists only one path $\pi[a,b]$ connecting them (so, there are no cycles in Ω). In this paper, we deal mainly with trees; however, some of the results concern the general case.

2.2. Optical metric

A function ρ defined on $\Omega\backslash V$ is said to be a *density* if

(i) ρ is strictly positive: $\rho(\cdot) \geqslant \rho_0 > 0$;
(ii) for each e, the function $\rho|_e$ may be extended on $\bar{e}$ up to a function of the class $C^2(\bar{e})$.

Thus, the boundary values $\rho(\gamma), \gamma \in \Gamma$ are well defined whereas at interior vertices $v \in V\backslash\Gamma$ the density may be considered as an $r(v)$-valued function.

Let $|\mathrm{d}x|^2$ be the Euclidean length element induced on Ω by the $\mathbf{R}^2$-metric. The *optical metric* τ on Ω is defined by the element

$$\mathrm{d}\tau^2 := \rho(x)\,|\mathrm{d}x|^2, \qquad x \in \Omega\backslash V; \tag{2.1}$$

by the properties of ρ this metric is equivalent to the Euclidean one. If Ω is a tree, it follows from (2.1) that

$$\tau(a,b) = \begin{cases} \int_{\pi[a,b]} \rho^{1/2}\,|\mathrm{d}x|, & a \neq b, \\ 0, & a = b. \end{cases}$$

Let us set

$$B_r[a] := \{x \in \Omega \mid \tau(a,x) < r\};$$

for a subset $A \subset \Omega$ we denote by

$$d(A) := \sup_{a,b \in A} \tau(a,b),$$

its (optical) diameter.

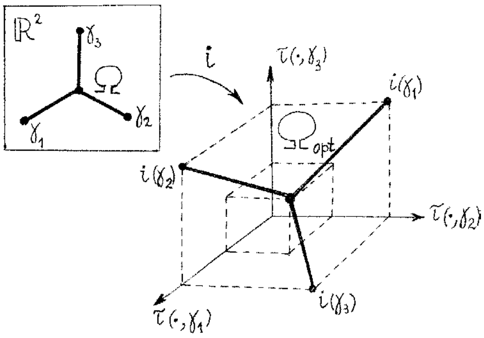

Figure 1. Optical image.

2.3. Optical model. Spatial isometry

Let us set $n := \sharp\Gamma$, numerate the boundary vertices and introduce a map $i : \Omega \mapsto \mathbf{R}^n$,

$$i(x) := \mathrm{col}\{\tau(x,\gamma_1),\ldots,\tau(x,\gamma_n)\}.$$

The set

$$\Omega_{\rm opt} := i(\Omega),$$

is a graph in $\mathbf{R}^n$ which we call the *optical image* of Ω. Endow $\Omega_{\rm opt}$ with a metric $\tau_{\rm opt}$ defined by the element

$$\mathrm{d}\tau_{\rm opt}{}^2 := \frac{1}{n}\|\mathrm{d}t\|^2_{\mathbf{R}^n}.$$

Lemma 1. *If Ω is a tree, the map i is an isometry of the metric space (Ω,τ) onto $(\Omega_{\rm opt},\tau_{\rm opt})$. The spaces $\left(\Omega_{\rm opt},\|\mathrm{d}t\|^2_{\mathbf{R}^n}\right)$ and $(\Omega,\ |\mathrm{d}x|^2)$ are homeomorphic.*

Proof. Show that the map i is injective. Assume a, b to be such that $a \neq b$ and $i(a) = i(b)$; as is evident, neither a nor b belongs to the boundary Γ. Since Ω is a tree, one can take $\gamma \in \Gamma$ such that $\pi[\gamma,a)\cap\pi[a,b] = \{\emptyset\}$, whereas for the path $\pi[\gamma,b] = \pi[\gamma,a)\cup\pi[a,b]$ one has

$$\tau(\gamma,b) = \tau(\gamma,a)+\tau(a,b) > \tau(\gamma,a).$$

On the other hand, $i(a) = i(b)$ implies $\tau(\gamma,a) = \tau(\gamma,b)$. Hence, the assumption $a \neq b, i(a) = i(b)$ leads to a contradiction.

Show that i is an isometry. Indeed, if a lies close enough to b (so that a and b belong to one and the same edge of the tree) then $|\tau(\gamma,a)-\tau(\gamma,b)| = \tau(a,b)$ for any $\gamma\in\Gamma$ that yields

$$\tau_{\rm opt}{}^2(i(a),i(b)) = \frac{1}{n}\sum_{j=1}^{n}|\tau(\gamma_j,a)-\tau(\gamma_j,b)|^2 = \tau^2(a,b).$$

Hence, i preserves the distance.

Since Euclidean and optical metrics are equivalent on Ω, the spaces $(\Omega,\ |\mathrm{d}x|^2)$ and (Ω,τ) are homeomorphic (through the map id) whereas i is a homeomorphism from the latter space onto $\left(\Omega_{\rm opt},\|\mathrm{d}t\|^2_{\mathbf{R}^n}\right)$. Therefore $(\Omega,\ |\mathrm{d}x|^2)$ is homeomorphic to $\left(\Omega_{\rm opt},\|\mathrm{d}t\|^2_{\mathbf{R}^n}\right)$. $\square$

Thus, the optical image of the tree determines its topology.

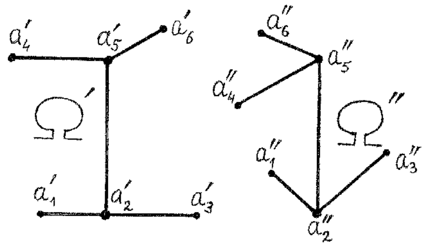

Figure 2. Spatially isometric graphs: $|a_i'a_j'| = |a_i''a_j''|$; $\rho' \equiv 1$, $\rho'' \equiv 1$.

One more consequence of the lemma is that the functions $\{\tau(\gamma_j, \cdot)\}_{j=1}^n$ form a coordinate system on a tree. We call $\tau(\gamma_j, \cdot)$ the *optical coordinates*.

Define the function

$$\rho_{\text{opt}} := \rho \circ i^{-1},$$

on Ω_{opt}. The pair $\Omega_{\text{opt}}, \rho_{\text{opt}}$ determines the original Ω, ρ (that is the subject of the inverse problem which we are going to solve) up to a natural nonuniqueness in the following sense.

We say that two graphs Ω' and Ω'' equipped with densities ρ' and ρ'' are *spatially isometric* if there exists an isometry I between the metric spaces $(\Omega', |\mathrm{d}x|^2)$ and $(\Omega'', |\mathrm{d}x|^2)$ such that $I(\Omega') = \Omega''$ and $\rho' = \rho'' \circ I$.

A simple fact is that a spatial isometry of Ω' and Ω'' implies $\Omega'_{\text{opt}} = \Omega''_{\text{opt}}$ and $\rho'_{\text{opt}} = \rho''_{\text{opt}}$. The converse also turns out to be true.

Lemma 2. *Let Ω', Ω'' be two graphs equipped with densities ρ', ρ''; assume that each of them is bijective to its optical image. If $\Omega'_{\text{opt}} = \Omega''_{\text{opt}}$ and $\rho'_{\text{opt}} = \rho''_{\text{opt}}$ then Ω' and Ω'' are spatially isometric.*

Proof. Since

$$i'(\Omega') = \Omega'_{\text{opt}} = \Omega''_{\text{opt}} = i''(\Omega''),$$

the map $I := (i'')^{-1} i'$ is a bijection from Ω' onto Ω''.

Let $|\mathrm{d}x'|^2$ and $|\mathrm{d}x''|^2$ be the Euclidean length elements on Ω' and Ω''. In accordance with (2.1), the equality $\rho'_{\text{opt}} = \rho''_{\text{opt}}$ implies

$$|\mathrm{d}x'|^2 = \frac{\mathrm{d}\tau^2}{\rho'_{\text{opt}}} = \frac{\mathrm{d}\tau^2}{\rho''_{\text{opt}}} = |\mathrm{d}x''|^2;$$

hence, I is an isometry of the metric spaces $(\Omega', |\mathrm{d}x'|^2)$ and $(\Omega'', |\mathrm{d}x''|^2)$. The relation $\rho' = \rho'' \circ I$ is evident. □

So, if Ω is a tree the pair $\Omega_{\text{opt}}, \rho_{\text{opt}}$ determines Ω, ρ up to a spatial isometry.

A triple $\Omega_{\text{opt}}, \tau_{\text{opt}}, \rho_{\text{opt}}$ is called the *optical model* of the original Ω, τ, ρ. In a clear sense, the optical model is an isometric copy of the equipped tree. A reason for introducing this concept is that, as we will see, it is $\Omega_{\text{opt}}, \tau_{\text{opt}}, \rho_{\text{opt}}$ which may be recovered from inverse data. The optical model, in its turn, determines the tree up to a spatial isometry.

2.4. Operator L and spectral data

Introduce the (real) Hilbert space $\mathcal{H} = L_{2,\rho}(\Omega)$,

$$(y, w)_{\mathcal{H}} := \int_\Omega yw\, \rho|\mathrm{d}x| = \sum_{e\in E} \int_e yw\, \rho|\mathrm{d}x|.$$

We assign a function y on Ω to the class $\mathcal{H}_0^1$ if

(i) $y \in C(\Omega)$;
(ii) $y|_{\bar{e}}$ belongs to the Sobolev class $H^1(\bar{e})$ on every edge e;
(iii) $y = 0$ on Γ,
and denote

$$(y, w)_{\mathcal{H}_0^1} := \int_\Omega \nabla y \cdot \nabla w |\mathrm{d}x| = \sum_{e \in E} \int_e \frac{\mathrm{d}y}{\mathrm{d}e} \frac{\mathrm{d}w}{\mathrm{d}e} |\mathrm{d}x|,$$

where ∇ is the Euclidean gradient on Ω, $\frac{\mathrm{d}}{\mathrm{d}e}$ is the differentiation with respect to the Euclidean length in any (but one and the same for y and w) direction along e.

Let us recall the well-known facts. The product $(\cdot\,,\cdot)_{\mathcal{H}_0^1}$ is a closed positive definite bilinear form in $\mathcal{H}$ defined on $\mathcal{H}_0^1 \times \mathcal{H}_0^1$. Such a form determines a (unique) self-adjoint operator $L : \mathcal{H} \mapsto \mathcal{H}$ such that $\operatorname{Dom} L \subset \mathcal{H}_0^1$ and

$$(Ly, w)_{\mathcal{H}} = (y, w)_{\mathcal{H}_0^1},$$

for $y \in \operatorname{Dom} L$, $w \in \mathcal{H}_0^1$ (see, e.g., [9]). This operator may be described in more detail as follows.

A function $y \in \mathcal{H}$ belongs to $\operatorname{Dom} L$ iff

(iv) $y \in \mathcal{H}_0^1$;
(v) $y|_{\bar{e}} \in H^2(\bar{e})$ on every $e \in E$;
(vi) for each interior vertex $v \in V \backslash \Gamma$ the equality (Kirchhoff's law)

$$\sum_{e \sim v} \frac{\mathrm{d}y}{\mathrm{d}e}(v) = 0, \tag{2.2}$$

holds where

$$\frac{\mathrm{d}y}{\mathrm{d}e}(v) := \lim_{x \in e, x \to v} \frac{y(x) - y(v)}{|x - v|}.$$

Furthermore, for $y \in \operatorname{Dom} L$ and $e \in E$ one has

$$(Ly)|_e = -\frac{1}{\rho} \frac{\mathrm{d}^2 y}{\mathrm{d}e^2} \tag{2.3}$$

(see, e.g., [11]).

In what follows we also use the operator $L_{\max} \supset L$ defined on continuous functions satisfying (v), (vi) and acting by the same rule (2.3).

The operator L has a discrete spectrum $\{\lambda_k\}_{k=1}^\infty : 0 < \lambda_1 < \lambda_2 \leqslant \cdots$ (see, e.g., [11, 13]); let $\{\varphi_k\}_{k=1}^\infty$ be the corresponding eigenfunctions normalized by $(\varphi_k, \varphi_l)_{\mathcal{H}} = \delta_{kl}$. The eigenvalue λ_1 is ordinary (of multiplicity 1); the eigenfunction φ_1 does not vanish in $\operatorname{int} \Omega$ (see, e.g., [16]) and in throughout what follows we assume

$$\varphi_1 > 0 \qquad \text{in} \quad \operatorname{int} \Omega. \tag{2.4}$$

For each boundary vertex $\gamma \in \Gamma$ the value

$$\frac{\mathrm{d}\varphi_k}{\mathrm{d}e}(\gamma) = \lim_{x \to \gamma} \frac{\varphi_k(x) - \varphi_k(\gamma)}{|x - \gamma|},$$

is well defined and we denote

$$\left.\frac{\mathrm{d}\varphi_k}{\mathrm{d}e}\right|_\Gamma := \operatorname{col}\left\{\frac{\mathrm{d}\varphi_k}{\mathrm{d}e}(\gamma_1), \ldots, \frac{\mathrm{d}\varphi_k}{\mathrm{d}e}(\gamma_n)\right\}.$$

So, with each equipped graph Ω one associates a set of pairs $\left\{\lambda_k;\, \frac{\mathrm{d}\varphi_k}{\mathrm{d}e}\Big|_\Gamma\right\}_{k=1}^{\infty}$ which is called the (Dirichlet) *spectral data*. It is the set which will play the role of data in the inverse problem.

As was already mentioned in the introduction, spatially isometric graphs have one and the same spectral data. Indeed, if $I : \Omega' \mapsto \Omega''$ is a spatial isometry one can easily check that the map $\hat{I} : \mathcal{H}' \mapsto \mathcal{H}''$, $\hat{I}y := y \circ I^{-1}$ is a unitary operator which provides $\hat{I}L' = L''\hat{I}$ (yielding $\lambda'_k = \lambda''_k$) and $\hat{I}\varphi'_k = \varphi''_k$ (implying $\frac{\mathrm{d}\varphi'_k}{\mathrm{d}e'}(\gamma'_j) = \frac{\mathrm{d}\varphi''_k}{\mathrm{d}e''}(\gamma''_j)$ where $\gamma''_j = I\gamma'_j$). Therefore, setting the goal of recovering a graph from its spectral data, what the boundary spectral inverse problem is, one can hope for determination only up to a spatial isometry. Theorem 1 claims that in the case of trees such a determination is possible. This fact is the main result of our paper; its proof is postponed until section 5.

2.5. Dirac measures

Let us introduce a triple of spaces (the rigged Hilbert space)

$$\mathcal{H}_0^1 \subset \mathcal{H} \subset \mathcal{H}^{-1},$$

where $\mathcal{H}^{-1}$ is dual to $\mathcal{H}_0^1$ with respect to the product $(\cdot\,,\cdot)_\mathcal{H}$. As is customary, we denote by $(h, y)_\mathcal{H}$ the value of a functional $h \in \mathcal{H}^{-1}$ on an element $y \in \mathcal{H}_0^1$. For each nonzero functional $h \in \mathcal{H}^{-1}$ we define its *optical diameter*

$$d[h] := d(\operatorname{supp} h).$$

Let

$$\mathcal{D} := \{c\delta_\xi \mid \xi \in \operatorname{int}\Omega,\, 0 \neq c \in \mathbf{R}\} \subset \mathcal{H}^{-1},$$

be the subset of functionals proportional to Dirac measures defined by

$$(\delta_\xi, y)_\mathcal{H} = y(\xi).$$

Denote by

$$\mathcal{D}_1 := \{h \in \mathcal{D} \mid \|h\|_{\mathcal{H}^{-1}} = 1, (h, \varphi_1)_\mathcal{H} > 0\},$$

the set of normalized Dirac measures; by virtue of agreement (2.4) all of them are positive. We omit the proof of the following simple facts.

Proposition 1. *The inclusion $h \in \mathcal{D}$ is equivalent to the relation $d[h] = 0$. The representation*

$$\mathcal{D}_1 = \{h \in \mathcal{H}^{-1} \mid \|h\|_{\mathcal{H}^{-1}} = 1, (h, \varphi_1)_\mathcal{H} > 0, d[h] = 0\}, \tag{2.5}$$

is valid.

Each Dirac measure is supported at a single interior point of the graph and later, solving the inverse problem, we will use an evident fact: the set $\mathcal{D}_1$ is bijective to $\operatorname{int}\Omega$ through the map $h \mapsto \operatorname{supp} h$.

3. Waves on graph

3.1. Dynamical system

Speaking about the dynamical system associated with a graph Ω, we mean an initial boundary value problem of the form

$$u_{tt} + L_{\max} u = 0 \qquad \text{in} \quad \operatorname{int}\Omega \times (0, T), \tag{3.1}$$

$$u|_{t=0} = u_t|_{t=0} = 0 \qquad \text{in} \quad \Omega, \tag{3.2}$$

$$u = f \qquad \text{on} \quad \Gamma \times [0, T], \tag{3.3}$$

with a (Dirichlet) boundary control $f = f(\gamma, t)$; $u = u^f(x, t)$ is a solution which describes a wave initiated at Γ and propagating in Ω. If f is a C^2-smooth function of t and satisfies $f(\cdot\,, 0) = f_t(\cdot\,, 0) = f_{tt}(\cdot\,, 0)$, the problem has the unique classical solution satisfying the wave equation

$$\rho u_{tt}(\cdot\,, t) - u_{ee}(\cdot\,, t) = 0,$$

on edges and the Kirchhoff law (2.2) at interior vertices for all times.

The space of controls $\mathcal{F}^T := L_2(\Gamma \times [0, T])$ with the inner product

$$(f, g)_{\mathcal{F}^T} = \sum_{\gamma \in \Gamma} \int_0^T f(\gamma, t) g(\gamma, t)\, \mathrm{d}t,$$

is called the *outer space* of the system (3.1)–(3.3). The space $\mathcal{H}$ is the *inner space*; the waves (states) $u^f(\cdot\,, t)$ are time-dependent elements of $\mathcal{H}$.

3.2. Fundamental solution

Fix $\gamma \in \Gamma$ and denote

$$\delta_\gamma(\gamma') := \begin{cases} 0, & \gamma' \neq \gamma, \\ 1, & \gamma' = \gamma. \end{cases}$$

Take $\chi \in C^2[0, T]$ such that $\chi \geqslant 0$, $\operatorname{supp} \chi \subset (0, T)$, and $\int_0^T \chi(t)\, \mathrm{d}t = 1$. Choose a small $\varepsilon > 0$ and denote

$$\chi_\varepsilon(t) := \begin{cases} \frac{1}{\varepsilon} \chi\left(\frac{t}{\varepsilon}\right), & 0 \leqslant t \leqslant \varepsilon T, \\ 0, & \varepsilon T < t < T; \end{cases}$$

let $u^{\delta_\gamma \chi_\varepsilon}$ be the classical solution to (3.1)–(3.3) corresponding to the control $f(\gamma', t) = \delta_\gamma(\gamma') \chi_\varepsilon(t)$. As may be shown, for $\varepsilon \to 0$ (so that χ_ε tends to the Dirac δ-function) the limit passage gives a function

$$u^{\delta_\gamma \delta} := \lim_{\varepsilon \to 0} u^{\delta_\gamma \chi_\varepsilon} \in C([0, T]; \mathcal{H}^{-1}),$$

which is the *fundamental solution* of the problem (3.1)–(3.3). This solution describes a wave initiated by an instantaneous source supported at γ. In the next section some of the properties of $u^{\delta_\gamma \delta}$ are presented in more detail.

3.3. Propagation of singularities

All the results of sections 3.3 and 3.4 are of general character: we do not assume Ω to be a tree. Recall that $\tau(a, b)$ is the optical distance between $a, b \in \Omega$.

First edge. Let e be the edge incident to $\gamma \in \Gamma$, $v \in V$ the second vertex incident to e. For times $0 < t \leqslant \tau(\gamma, v)$ the well-known representation (time-domain WKB expansion)

$$u^{\delta_\gamma \delta}(x, t) = \left[\frac{\rho(x)}{\rho(\gamma)}\right]^{-\frac{1}{4}} \delta(t - \tau(x, \gamma)) + u_\gamma(x, t), \qquad x \in \Omega, \tag{3.4}$$

holds, where u_γ is a bounded function satisfying $\operatorname{supp} u_\gamma(\cdot\,, t) \subseteq \overline{B_t[\gamma]} \subseteq \bar{e}$ and C^2-smooth on $[\operatorname{supp} u_\gamma] \backslash \{(\gamma, 0)\}$ (see, e.g., [10]). The first term in (3.4) is interpreted as a leading singularity

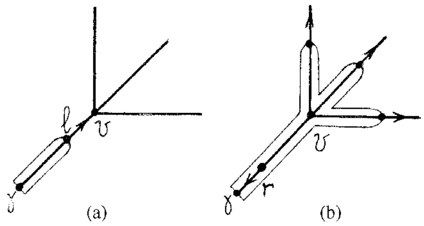

Figure 3. Passing through v: (a) $t = \tau(\gamma, v) - \varepsilon$; (b) $t = \tau(\gamma, v) + \varepsilon$.

which moves from γ towards v along e with a variable speed $\rho^{-1/2}$: see the point l in figure 3(a) (the ball $B_t[\gamma]$ is contoured).

Passing through the first vertex. At the moment $t = \tau(\gamma, v)$ the singularity reaches v and, then, passes through v. A simple analysis using (3.4) and (2.3) gives the following representation for times $\tau(\gamma, v) < t \leqslant \tau(\gamma, v) + \varepsilon$ (with small $\varepsilon > 0$):

$$u^{\delta_\gamma \delta}(x, t) = \alpha(x)\delta(t - \tau(\gamma, v) - \tau(v, x)) + u_\gamma(x, t), \qquad x \in \Omega, \tag{3.5}$$

with a bounded u_γ satisfying $\operatorname{supp} u_\gamma(\cdot\,, t) \subseteq \overline{B_t[\gamma]}$ and piecewise smooth on $\operatorname{supp} u_\gamma$. The *amplitude* α is a piecewise smooth function strictly positive on edges $e' \neq e$ incident to v.

Note in addition that $\alpha|_e$ can be negative. From the physical point of view, this means that the reflected part of the singularity moving back to γ (point r in figure 3(b)) can have the phase opposite to the phase of the incident singularity (point l in figure 3(a); in both cases the ball $B_t[\gamma]$ is contoured).

First reflection from the boundary. Let $\gamma' \in \Gamma$ be a boundary vertex nearest to v:

$$\tau(\gamma', v) = \min_{\gamma'' \in \Gamma} \tau(\gamma'', v)$$

(may be $\gamma = \gamma'$). As $t \to \tau(\gamma, v) + \tau(v, \gamma') - 0$, the singularity passing through v (and, perhaps, through another vertex or reflected from v back to γ) approaches γ'. Then the singularity is reflected by γ': see the point r in figure 3(b). A simple analysis using the condition

$$u^{\delta_\gamma \delta}(\cdot\,, t)|_\Gamma = 0, \qquad t > 0,$$

gives a representation

$$u^{\delta_\gamma \delta}(x, t) = \alpha(x)\delta(t - \tau(\gamma, v) - \tau(v, \gamma') - \tau(\gamma', x)) + u_\gamma(x, t), \tag{3.6}$$

with a bounded piecewise smooth u_γ which holds for t close to $\tau(\gamma, v) + \tau(v, \gamma') + \varepsilon$ and x close to γ'. The amplitude α is negative if $\gamma' \neq \gamma$ and positive if $\gamma' = \gamma$ which means that the reflection from the boundary leads to inverting the phase of the incident singularity.

Arbitrary times. Representations (3.4)–(3.6) are sufficient for further analysis of propagation of singularities in Ω. Omitting the proofs, we summarize the results in proposition 2 as follows.

In the spacetime domain $\Omega \times \bar{\mathbf{R}}_+$, for a fixed (x_0, t_0) let us define a *characteristic cone*

$$\mathcal{K}_{(x_0, t_0)} := \{(x, t) \mid \tau(x, x_0) = t - t_0\};$$

for a subset $A \subset \Omega \times \bar{\mathbf{R}}_+$ we denote

$$\mathcal{K}_A := \bigcup_{(x,t) \in A} \mathcal{K}_{(x,t)}.$$

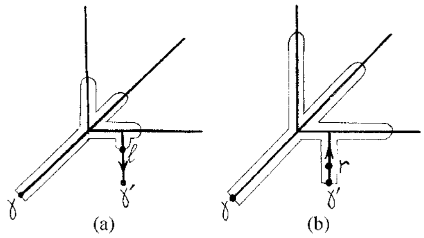

Figure 4. Reflection from $\gamma \neq \gamma'$: (a) $t = \tau(\gamma, v) + \tau(v, \gamma') - \varepsilon$; (b) $t = \tau(\gamma, v) + \tau(v, \gamma') + \varepsilon$.

Introduce also a *characteristic set* $\mathcal{C}_{(x_0,t_0)}$ which is defined by the following recurrent procedure:

Step 0. Put

$$\mathcal{C}^{[0]}_{(x_0,t_0)} := \mathcal{K}_{(x_0,t_0)},$$

and

$$\mathcal{V}^{[0]}_{(x_0,t_0)} := \left\{(x, t) \in \mathcal{C}^{[0]}_{(x_0,t_0)} \,\middle|\, x \in V\right\};$$
$$\ldots.$$

Step j. Put

$$\mathcal{C}^{[j]}_{(x_0,t_0)} := \mathcal{C}^{[j-1]}_{(x_0,t_0)} \bigcup \mathcal{K}_{\mathcal{V}^{[j-i]}_{(x_0,t_0)}},$$

and

$$\mathcal{V}^{[j]}_{(x_0,t_0)} := \left\{(x, t) \in \mathcal{C}^{[j-1]}_{(x_0,t_0)} \,\middle|\, x \in V\right\};$$
$$\ldots;$$

define

$$\mathcal{C}_{(x_0,t_0)} := \bigcup_{j=0}^{\infty} \mathcal{C}^{[j]}_{(x_0,t_0)}.$$

Note that $\mathcal{C}_{(x_0,t_0)}$ can be also characterized as a minimal set in $\Omega \times \bar{\mathbf{R}}_+$ satisfying the conditions:

(i) $\mathcal{K}_{(x_0,t_0)} \subset \mathcal{C}_{(x_0,t_0)}$;
(ii) if $v \in V$ and $t_v \in \mathbf{R}_+$ are such that $(v, t_v) \in \mathcal{C}_{(x_0,t_0)}$ then $\mathcal{K}_{(v,t_v)} \subset \mathcal{C}_{(x_0,t_0)}$.

This set is of the form of a spacetime graph; figure 5 illustrates the case of $x_0 = \gamma$, $t_0 = 0$.

We also denote by

$$\mathcal{K}^s_{(x_0,t_0)} := \left\{x \in \Omega \,\middle|\, (x, s) \in \mathcal{K}_{(x_0,t_0)}\right\}, \qquad \mathcal{C}^s_{(x_0,t_0)} := \left\{x \in \Omega \,\middle|\, (x, s) \in \mathcal{C}_{(x_0,t_0)}\right\}$$

the cross-sections by the plane $t = s$; as is easy to see, one has

$$\mathcal{K}^s_{(x_0,t_0)} = \{x \in \Omega \mid \tau(x, x_0) = s - t_0\}.$$

Proposition 2. *The leading singularities of* $u^{\delta_\gamma \delta}$ *(as a spacetime distribution) are located on the characteristic set* $\mathcal{C}_{(\gamma,0)}$. *For any* $t > 0$ *a representation*

$$u^{\delta_\gamma \delta}(\cdot\,, t) = \sum_{\xi \in \mathcal{K}^t_{(\gamma,0)}} \alpha_\gamma(\xi)\delta_\xi(\cdot) + \sum_{\xi \in \mathcal{C}^t_{(\gamma,0)} \setminus \mathcal{K}^t_{(\gamma,0)}} \beta_\gamma(\xi, t)\delta_\xi(\cdot) + u_\gamma(\cdot\,, t), \tag{3.7}$$

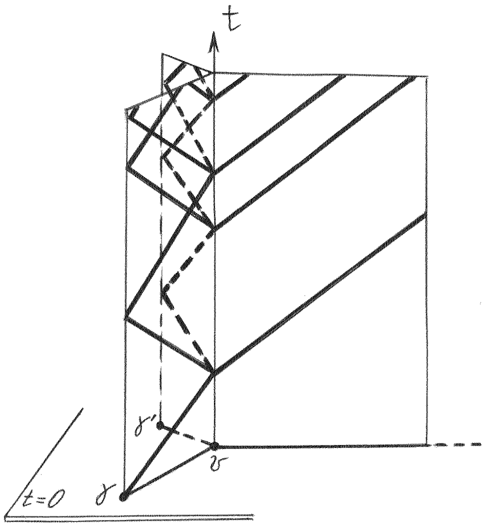

Figure 5. The set $\mathcal{C}_{(\gamma,0)}$.

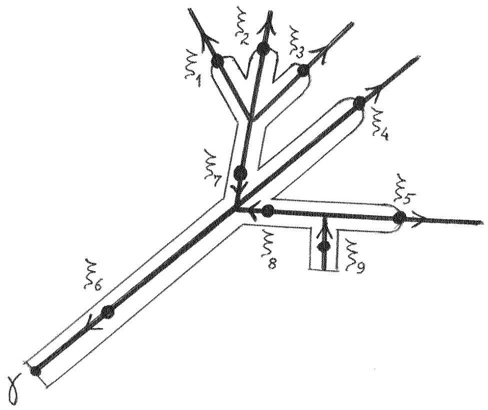

Figure 6. Singularities of $u^{\delta_\gamma\delta}(\cdot\,,t)$.

holds in Ω *where:*

$\delta_\xi \in \mathcal{H}^{-1}$ *is the Dirac measure supported at* ξ*;*
the amplitude function α_γ *is strictly positive on* Ω*, piecewise smooth and* C^2*-smooth out of the interior vertices; the value* $\alpha_\gamma(\xi)$ *is determined by the density* $\rho|_{\pi[\gamma,\xi]}$*;*
β_γ *is a piecewise smooth function defined on* $\mathcal{C}_{(\gamma,0)}\backslash\mathcal{K}_{(\gamma,0)}$*;*
$u_\gamma(\cdot\,,t)$ *is a bounded piecewise smooth function (with possible jumps on* $\mathcal{C}^t_{(\gamma,0)}$*) satisfying* $\operatorname{supp} u_\gamma(\cdot\,,t) \subset \overline{B_t[\gamma]}$.

The first sum in (3.7) selects the so-called forward singularities which lie on the forward front of the wave $u^{\delta_\gamma\delta}(\cdot\,,t)$ moving from γ. These singularities are present only if $t < \max_{x\in\Omega}\tau(x,\gamma)$ whereas $\alpha_\gamma(\xi)$ is the amplitude of the *first* singularity arriving at ξ from γ at the moment $t = \tau(\xi,\gamma)$.

The second sum is the *secondary singularities* appearing due to reflection from vertices and supported into $B_t[\gamma]$.

A typical picture is shown in figure 6 where the points $\xi_1,\ldots,\xi_5$ support the forward singularities whereas $\xi_6,\ldots,\xi_9$ support the secondary ones (the ball $B_t[\gamma]$ is contoured).

3.4. Generalized solutions

Here we list some of the well-known properties of the solutions to the problem (3.1)–(3.3) which may be easily derived from the representation (3.7).

Let

$$\mathcal{F}^T_\gamma := \{f \in \mathcal{F}^T \mid \operatorname{supp} f \subset \gamma \times [0,T]\},$$

be the subspace of controls acting from $\gamma \in \Gamma$. Introduce the classes

$$(\mathcal{F}^T)^1_0 := \left\{ f \in \mathcal{F}^T \,\middle|\, f(\cdot\,,t) = \sum_{\gamma\in\Gamma} \delta_\gamma(\cdot) g_\gamma(t), \quad g_\gamma \in H^1_0[0,T] \right\}$$

(so that the controls belonging to this class satisfy $f(\cdot,0)=f(\cdot,T)=0$) and

$$\left(\mathcal{F}^T_\gamma\right)^1_0 := (\mathcal{F}^T)^1_0 \bigcap \mathcal{F}^T_\gamma .$$

Let $\mathcal{H}^1 := H^1(\Omega) \supset \mathcal{H}^1_0$ be the Sobolev class of functions on Ω satisfying conditions (i), (ii), section 2.4.

(i) *Duhamel formula*. If f is such that u^f is classical, a representation

$$u^f = \sum_{\gamma\in\Gamma} u^{\delta_\gamma\delta} * f(\gamma,\cdot) \tag{3.8}$$

(the convolution with respect to time) holds.

(ii) *Generalized solutions*. For any $f \in \mathcal{F}^T$ the right-hand side of (3.8) is well defined and belongs to the class $C([0,T];\mathcal{H})$; it defines a solution u^f to (3.1)–(3.3) of this class.

(iii) *Localization of solutions*. For controls $f \in \mathcal{F}^T_\gamma$ the inclusion

$$\operatorname{supp} u^f(\cdot,t) \subset \overline{B_t[\gamma]}, \qquad t \geqslant 0, \tag{3.9}$$

holds.

(iv) *Delayed solutions*. For $f \in \mathcal{F}^T$ denote

$$f_s(\cdot,t) := \begin{cases} 0, & 0 \leqslant t < s,\\ f(\cdot,t-s), & s \leqslant t \leqslant T. \end{cases}$$

Since the graph and its density do not depend on time, one has

$$u^{f_s}(\cdot,t) = u^f(\cdot,t-s).$$

As a result, if $f(\cdot,t)=0$ for $0 \leqslant t \leqslant s$, one has

$$\operatorname{supp} u^f(\cdot,t) \subset \overline{B_{t-s}[\gamma]}, \qquad t \geqslant 0. \tag{3.10}$$

(v) *Smoothness*. In the case where $f \in (\mathcal{F}^T)^1_0$ one has $u^f(\cdot,t) \in \mathcal{H}^1$ for all $t \in [0,T]$ and $u^f(\cdot,T) \in \mathcal{H}^1_0$.

3.5. Propagators p_γ

Here we introduce the solutions (waves) of a special kind. Playing the role of a device prospecting the graph from its boundary, these waves will be invoked for solving the inverse problem. In sections 3.5 and 3.6 we also do not assume Ω to be a tree.

Let

$$\theta(t) := \begin{cases} 0, & t < 0,\\ 1, & t \geqslant 0, \end{cases}$$

be the Heaviside function; putting in (3.1)–(3.3) $f(\gamma',t) = \delta_\gamma(\gamma')\theta(t)$ we say the solution

$$p_\gamma := u^{\delta_\gamma\theta} = u^{\delta_\gamma\delta} * \theta,$$

to be the *propagator* corresponding to a boundary vertex $\gamma \in \Gamma$. The propagator is a function of the class $C([0,T];\mathcal{H})$ satisfying (3.9).

Assume the final moment T to be large enough: $T > d(\Omega)$ and take $t < \max_{x\in\Omega}\tau(x,\gamma)$, so that $\mathcal{K}^t_{(\gamma,0)} \neq \{\emptyset\}$. Fix a $\xi \in \mathcal{K}^t_{(\gamma,0)}$ (see points $\xi_1,\ldots,\xi_5$ on figure 6) and note the relation $t = \tau(\gamma,\xi)$. Using (3.7), one can easily derive a representation

$$p_\gamma(\cdot,t) = \alpha_\gamma(\xi)\theta(t-\tau(\cdot,\gamma)) + w_\gamma(\cdot,t), \tag{3.11}$$

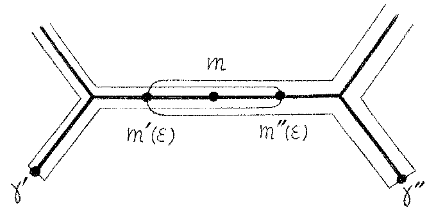

Figure 7. Interaction of propagators.

which is valid in $B_\varepsilon[\xi]$ with a small enough $\varepsilon > 0$ and a continuous w_γ vanishing at points x satisfying $\tau(x, \gamma) \geqslant t$ (in particular, $w(\xi, t) = 0$). Thus, the propagator p_γ carries a jump on its forward front.

Choose $\gamma', \gamma'' \in \Gamma$; the value

$$t_{\gamma'\gamma''} := \inf\{t > 0 \mid (p_{\gamma'}(\cdot\,, t), p_{\gamma''}(\cdot\,, t))_{\mathcal{H}} \neq 0\},$$

may be interpreted as the time at which propagators $p_{\gamma'}$ and $p_{\gamma''}$ begin to interact.

Lemma 3. *The equality*

$$t_{\gamma'\gamma''} = \frac{\tau(\gamma', \gamma'')}{2}, \tag{3.12}$$

is valid.

Proof. Let us deal with the nontrivial case of $\gamma' \neq \gamma''$. Let $m \in \Omega$ be the middle point of a shortest path $\pi[\gamma', \gamma'']$ connecting γ' with γ''; denote

$$T_m := \tau(\gamma', m) = \tau(m, \gamma'') = \frac{\tau(\gamma', \gamma'')}{2}.$$

Consider the generic case $m \notin V$; the case of $m \in V$ may be studied quite analogously. So, let m lie on an edge e.

If $t < T_m$, by virtue of (3.9) one has

$$(p_{\gamma'}(\cdot\,, t), p_{\gamma''}(\cdot\,, t))_{\mathcal{H}} = \int_{B_t[\gamma'] \cap B_t[\gamma'']} p_{\gamma'}(\cdot\,, t) p_{\gamma''}(\cdot\,, t) \rho\, |\mathrm{d}x| = 0,$$

because the balls do not intersect. Hence, we have $t_{\gamma'\gamma''} \geqslant T_m$.

Take $\varepsilon > 0$ so small that

$$B_{T_m+\varepsilon}[\gamma'] \bigcap B_{T_m+\varepsilon}[\gamma''] \subset e;$$

let $m'(\varepsilon)$ and $m''(\varepsilon)$ be the endpoints of this intersection nearest to γ' and γ'' respectively (see figure 7; the balls $B_{T_m+\varepsilon}[\gamma']$ and $B_{T_m+\varepsilon}[\gamma'']$ are contoured).

In accordance with (3.11) (for $t = T_m + \varepsilon$), on the segment $\pi[m'(\varepsilon), m''(\varepsilon)]$ one has the representations

$$p_{\gamma'}(\cdot\,, T_m + \varepsilon) = \alpha_{\gamma'}(m''(\varepsilon))\theta(T_m + \varepsilon - \tau(\cdot\,, \gamma')) + w_{\gamma'}(\cdot\,, T_m + \varepsilon),$$

with a continuous bounded $w_{\gamma'}$ satisfying $w_{\gamma'}(m''(\varepsilon), T_m + \varepsilon) = 0$ and

$$p_{\gamma''}(\cdot\,, T_m + \varepsilon) = \alpha_{\gamma''}(m'(\varepsilon))\theta(T_m + \varepsilon - \tau(\cdot\,, \gamma'')) + w_{\gamma''}(\cdot\,, T_m + \varepsilon),$$

with a continuous bounded $w_{\gamma''}$ satisfying $w_{\gamma''}(m'(\varepsilon), T_m + \varepsilon) = 0$. For $\varepsilon \to 0$, these representations easily lead to the asymptotics

$$(p_{\gamma'}(\cdot\,, T_m + \varepsilon), p_{\gamma''}(\cdot\,, T_m + \varepsilon))_{\mathcal{H}} = \alpha_{\gamma'}(m)\alpha_{\gamma''}(m) \left(\int_{B_\varepsilon[m]} \rho |\mathrm{d}x| \right) [1 + o(1)],$$

which implies $(p_{\gamma'}(\cdot\,, T_m + \varepsilon), p_{\gamma''}(\cdot\,, T_m + \varepsilon))_{\mathcal{H}} \neq 0$ for small enough $\varepsilon > 0$. Whence, $t_{\gamma'\gamma''} \leqslant T_m + \varepsilon$ and, finally, $t_{\gamma'\gamma''} \leqslant T_m$ by arbitrariness of ε. As a result, we obtain $t_{\gamma'\gamma''} = T_m$, i.e. (3.12) holds. □

3.6. Propagators p_ξ

Fix a $\xi \in \operatorname{int}\Omega$ and denote $c_\xi := \|\delta_\xi\|^{-1}_{\mathcal{H}^{-1}}$; the solution $p = p_\xi(x,t)$ to the problem

$$p_{tt} + Lp = 0 \qquad \text{in} \quad \operatorname{int}\Omega \times \mathbf{R}_+, \tag{3.13}$$

$$p|_{t=0} = 0, \qquad p_t|_{t=0} = c_\xi \delta_\xi \quad \text{in} \quad \Omega, \tag{3.14}$$

$$p = 0 \qquad \text{on} \quad \Gamma \times \bar{\mathbf{R}}_+, \tag{3.15}$$

is the *propagator* corresponding to an interior point ξ. The following facts may be easily established by means of a standard time-domain WKB technique (see, e.g., [10]) :

(i) $p_\xi \in C(\bar{\mathbf{R}}_+; \mathcal{H})$; supp $p_\xi(\cdot\,, t) \subset \overline{B_t[\xi]}$;

(ii) in the case of $\xi \notin V$, for times $0 < t < \tau(\xi, V)$ a representation

$$p_\xi(x,t) = \frac{c_\xi}{2}\left[\frac{\rho(x)}{\rho^3(\xi)}\right]^{-\frac{1}{4}} \theta(t - \tau(x,\xi)) + w_\xi(x,t), \qquad x \in \Omega, \tag{3.16}$$

holds with $w_\xi(\cdot\,, t) \in C(\Omega)$ (so that $w_\xi(\cdot\,, t) = 0$ at the endpoints of $\overline{B_t[\xi]}$);

(iii) for any $\xi \in \operatorname{int}\Omega$ and times $0 < t < \max_{x\in\Omega} \tau(x,\xi)$ a representation

$$p_\xi(\cdot\,, t) = c_\xi \alpha_\xi(\cdot)\theta(t - \tau(\cdot\,, \xi)) + w_\xi(\cdot\,, t), \tag{3.17}$$

is valid in Ω with a piecewise C^2-smooth strictly positive α_ξ and $w_\xi(\cdot\,, t)$ continuous near $\mathcal{K}^t_{(\xi,0)}$ (so that $w_\xi(\cdot\,, t) = 0$ on $\mathcal{K}^t_{(\xi,0)}$). Thus, the propagator p_ξ has a jump at its forward front; note in addition that another possible discontinuities (jumps) of p_ξ are located at the characteristic set $\mathcal{C}_{(\xi,0)}$.

Let $1(\cdot)$ be the function equal identically to 1 and $\xi \notin V$; for $t \to +0$, representation (3.16) easily gives the asymptotics

$$(p_\xi(\cdot\,, t), 1)_{\mathcal{H}} = c_\xi \rho(\xi) t + o(t), \qquad \|p_\xi(\cdot\,, t)\|^2_{\mathcal{H}} = \frac{c_\xi^2}{2}\rho^{3/2}(\xi) t + o(t),$$

leading to a useful relation for the density

$$\left\{\lim_{t\to 0} \frac{(p_\xi(\cdot\,, t), 1)^2_{\mathcal{H}}}{2t\|p_\xi(\cdot\,, t)\|^2_{\mathcal{H}}}\right\}^2 = \rho(\xi), \tag{3.18}$$

which will work in the inverse problem. Note in addition that in the case of $\xi \in V$ the left-hand side of (3.18) is equal to a weighted mean value of the limit values of the density along the edges incident to ξ.

3.7. Interaction times and optical distances

For $\xi', \xi'' \in \operatorname{int}\Omega$ and $\gamma \in \Gamma$ introduce the interaction times

$$t_{\xi'\xi''} := \inf\{t > 0 \mid (p_{\xi'}(\cdot\,, t), p_{\xi''}(\cdot\,, t))_{\mathcal{H}} \neq 0\},$$

and

$$t_{\gamma\xi} := \inf\{t > 0 \mid (p_\gamma(\cdot\,, t), p_\xi(\cdot\,, t))_{\mathcal{H}} \neq 0\}.$$

Representation (3.17) together with arguments quite analogous to those used in the proof of lemma 3 easily leads to the relations

$$t_{\xi'\xi''} = \frac{\tau(\xi', \xi'')}{2}, \qquad t_{\gamma\xi} = \frac{\tau(\gamma, \xi)}{2}, \tag{3.19}$$

and, so, we arrive at the following result generalizing (3.12) and (3.19).

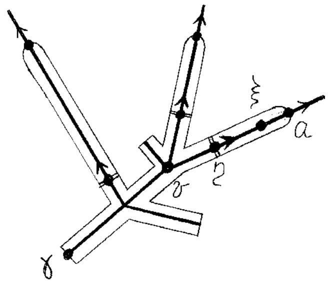

Figure 8. The neighbourhood of the forward front.

Proposition 3. *For any graph* Ω *and any two of its points* a, b *the equality*

$$t_{ab} = \frac{\tau(a, b)}{2},$$

is valid

4. Controllability

4.1. Wave shaping

Here we consider the question: Managing the boundary control, can one create waves of a prescribed shape? In rigorous formulation, this is a variant of the boundary control problem for the system (3.1)–(3.3): given a function $y \in \mathcal{H}$ to find a control $f \in \mathcal{F}^T$ providing

$$u^f(\cdot\,, T) = y \tag{4.1}$$

(see [2, 8, 15]).

This kind of problems is the subject of the boundary control theory which lies in the base of our approach and motivates the term 'BC method'. The approach follows a very general principle of the system theory: the richer the set of states of a dynamical system (in our case, the set of waves $u^f(\cdot\,, T)$) which an external observer can create by means of controls, the richer the information about this system which the observer can extract from external measurements (from inverse data). In particular, a possibility of creating the Dirac δ-functions in the interior points (that is a richest set of states because any state is a superposition of δ-functions) provides a way to solve inverse problems: see, e.g., [3].

The result of the next section details a character of controllability of the system (3.1)–(3.3). Note in advance that it is valid for any graphs, not only trees.

4.2. Local controllability near forward front

Fix $\gamma \in \Gamma$; let $T > 0$ be such that $\Omega \backslash \overline{B_T[\gamma]} \neq \{\emptyset\}$, so that the ball $\overline{B_T[\gamma]}$ is a proper subgraph of Ω. In this case the set

$$\mathcal{K}^T_{(\gamma,0)} = \{x \in \Omega \mid \tau(x, \gamma) = T\},$$

is nonempty; in accordance with (3.7) it supports the forward singularities of the fundamental solution $u^{\delta_\gamma\delta}(\cdot\,, T)$.

Choose any $a \in \mathcal{K}^T_{(\gamma,0)}$; let e be an edge of $\overline{B_T[\gamma]}$ incident to a (in figure 8 one has $e = \pi(a, v)$).

The point a supports the forward singularity $\alpha_\gamma(a)\delta_a(\cdot)$ of $u^{\delta_\gamma\delta}(\cdot\,, T)$; at the same time there exists an interval $e_a \subseteq e$ (see $\pi(a, \eta)$ in figure 8) which is free of secondary singularities.

The following result shows that e_a is a part of a neighbourhood of the forward front $\mathcal{K}^T_{(\gamma,0)}$ available for controlling from the vertex γ. Recall that $d(\cdot)$ is the optical diameter.

Lemma 4. *For any* $y \in \mathcal{H}$ *such that* $\operatorname{supp} y \subset \overline{e_a}$ *there exists the unique control* $f \in \mathcal{F}^T_\gamma$ *satisfying* $\operatorname{supp} f \subset \gamma \times [0, d(e_a)]$ *and providing the equality*

$$u^f(\cdot\,, T)|_{e_a} = y. \tag{4.2}$$

If, additionally, $y \in \mathcal{H}^1_0$, *one can find* $f \in \left(\mathcal{F}^T_\gamma\right)^1_0$ *such that (4.2) holds.*

Proof. Fix $\xi \in e_a$ and denote

$$f^\xi(\cdot\,, t) := \delta_\gamma(\cdot)\delta(t - \tau(\xi, a)).$$

Taking into account the delay $\tau(\xi, a)$ and using (3.7), one can easily represent

$$u^{f^\xi}(\cdot\,, T) = \alpha_\gamma(\xi)\delta_\xi(\cdot) + u^\xi(\cdot) \qquad \text{on} \quad e_a, \tag{4.3}$$

with C^2-smooth strictly positive α_γ and u^ξ satisfying

$$\operatorname{supp} u^\xi \subset \{x \in e_a \mid \tau(x, a) \geqslant \tau(x, \xi)\},$$

and C^2-smooth on its support.

Choose $y \in \mathcal{H}$ with $\operatorname{supp} y \subset \overline{e_a}$. Solving the control problem (4.2) with respect to f, let us seek for f in the form of a composition

$$f = \int_{e_a} f^\xi \psi(\xi)|\mathrm{d}\xi|, \tag{4.4}$$

which is a well-defined element of $\mathcal{F}^T$ for any $\psi \in L_2(e_a)$. By linearity of the map $f \mapsto u^f$, for $x \in e_a$ one has

$$\begin{aligned} y(x) = u^f(x, T) = \int_{e_a} u^{f^\xi}(x, T)\psi(\xi)|\mathrm{d}\xi| &= \langle \text{see (3.3)} \rangle \\ &= \alpha_\gamma(x)\psi(x) + \int_{\pi[x,a]} u^\xi(x)\psi(\xi)|\mathrm{d}\xi|. \end{aligned}$$

Parametrizing $x \in e_a$ by the optical distance $\tau = \tau(x, a)$ we obtain a Volterra equation of the form

$$\alpha_\gamma(\tau)\psi(\tau) + \int_0^\tau u^s(\tau)\psi(s)\frac{\mathrm{d}s}{\rho^{1/2}(s)} = y(\tau), \qquad 0 \leqslant \tau \leqslant d(e_a), \tag{4.5}$$

which is uniquely solvable in $L_2(0, d(e))$. Determining ψ and substituting in (4.4) we easily find

$$f(\cdot\,, t) = \delta_\gamma(\cdot)\int_0^{d(e_a)} \delta(t - \tau)\psi(\tau)\frac{\mathrm{d}\tau}{\rho^{1/2}(\tau)} = \delta_\gamma(\cdot)\frac{\psi(t)}{\rho^{1/2}(t)}, \qquad 0 \leqslant t \leqslant d(e_a). \tag{4.6}$$

At last, extending f onto $0 \leqslant t \leqslant T$ by zero, we obtain a control of the class $\mathcal{F}^T_\gamma$ mentioned in the statement of the lemma.

Due to a smoothness of α_γ, u^ξ and ρ, in the case of $y \in \mathcal{H}^1_0$ equation (4.5) gives $\psi \in H^1[0, d(e_a)]$. Since a lies at the boundary of $B_T[\gamma]$, one has $y(a) = 0$ that yields $\psi(0) = y(0) = 0$. Consequently, the control (4.6) satisfies $f \in H^1[0, d(e)]$ and $f(\gamma, 0) = 0$. Such a control may be extended (not uniquely) from $0 \leqslant t \leqslant d(e)$ onto $0 \leqslant t \leqslant T$ up to a function of the class $\left(\mathcal{F}^T_\gamma\right)^1_0$. This extension does not violate (4.2). Indeed, any two extensions f' and f'' of f can differ only in the interval $d(e_a) < t < T$; hence, by virtue of (3.10) the corresponding solutions $u^{f'}(\cdot\,, T)$ and $u^{f''}(\cdot\,, T)$ can differ only in $B_{T-d(e_a)}[\gamma]$ and must coincide on e_a. □

To be precise, the result of the lemma means that one can control not a neighbourhood of the forward front (separated by double lines in figure 8) *as a whole* but each of its components (such as e_a) *individually*. Indeed, by setting $y|_{e_a}$ one determines the control $f|_{[0,d(e_a)]}$ which, in its turn, determines the values of the wave $u^f(\cdot\,,T)$ everywhere on the set $\overline{B}_T[\gamma]\backslash B_{T-d(e_a)}[\gamma]$ containing a neigbourhood of the forward front. Therefore, being given an arbitrary y on $B_T[\gamma]$ one can get problem (4.1) unsolvable. See endnote 2

4.3. *Interaction 'wave functional'*

A formula derived in this section plays the key role in the inverse problem. It expresses the optical diameter of the functional in 'wave' terms. The derivation uses the local controllability established in lemma 2.

Fix a boundary vertex $\gamma \in \Gamma$; due to the smoothness property (v), section 3.4 the value

$$t_\gamma(h) := \inf\bigl\{T > 0 \,|\, \exists f \in \bigl(\mathcal{F}^T_\gamma\bigr)^1_0 :\ (h, u^f(\cdot\,,T))_{\mathcal{H}} \neq 0\bigr\},$$

is well-defined for functionals $h \in \mathcal{H}^{-1}$ and may be interpreted as a time at which the waves generated at γ begin to interact with the functional. If $(h, u^f(\cdot\,,T))_{\mathcal{H}} = 0$ for all T and f we set $t_\gamma(h) = +\infty$.

Taking $T < \tau(\gamma, \operatorname{supp} h)$, by virtue of the localization property (3.9) one has

$$\{\operatorname{supp} u^f(\cdot\,,T) \cap \operatorname{supp} h\} \subset \{\overline{B_T[\gamma]} \cap \operatorname{supp} h\} = \{\emptyset\},$$

which implies $(h, u^f(\cdot\,,T))_{\mathcal{H}} = 0$ and easily leads to

$$t_\gamma(h) \geqslant \tau(\gamma, \operatorname{supp} h). \tag{4.7}$$

At the same time, it is not difficult to show the examples realizing the strong inequality in (4.7).

Recall that $d[h] := d(\operatorname{supp} h)$; the following result is crucial for the inverse problem.

Lemma 5. *For any graph Ω the relation*

$$d[h] \geqslant \max_{\gamma',\gamma''\in\Gamma}\{2t_{\gamma'\gamma''} - [t_{\gamma'}(h) + t_{\gamma''}(h)]\}, \tag{4.8}$$

holds for all nonzero $h \in \mathcal{H}^{-1}$. If Ω is a tree, the equality

$$d[h] = \max_{\gamma',\gamma''\in\Gamma}\{2t_{\gamma'\gamma''} - [t_{\gamma'}(h) + t_{\gamma''}(h)]\}, \tag{4.9}$$

is valid.

Proof. (i) Choose a pair $\gamma', \gamma'' \in \Gamma$; let $b', b'' \in \operatorname{supp} h$ be such that $\tau(\gamma', b') = \tau(\gamma', \operatorname{supp} h)$ and $\tau(\gamma'', b'') = \tau(\gamma'', \operatorname{supp} h)$. The following relations are quite evident:

$$\begin{aligned}\tau(\gamma', \gamma'') &\leqslant \tau(\gamma', b') + \tau(b', b'') + \tau(\gamma'', b'')\\ &= \tau(\gamma', \operatorname{supp} h) + \tau(b', b'') + \tau(\gamma'', \operatorname{supp} h)\\ &\leqslant \tau(\gamma', \operatorname{supp} h) + d[h] + \tau(\gamma'', \operatorname{supp} h).\end{aligned} \tag{4.10}$$

Then we have

$$\begin{aligned}2t_{\gamma'\gamma''} &= \langle\text{see (3.12)}\rangle = \tau(\gamma', \gamma'') \leqslant \langle\text{see (4.10)}\rangle\\ &\leqslant \tau(\gamma', \operatorname{supp} h) + d[h] + \tau(\gamma'', \operatorname{supp} h) \leqslant \langle\text{see (4.7)}\rangle\\ &\leqslant t_{\gamma'}(h) + d[h] + t_{\gamma''}(h);\end{aligned}$$

hence, one has

$$d[h] \geqslant 2t_{\gamma'\gamma''} - [t_{\gamma'}(h) + t_{\gamma''}(h)].$$

Taking the maximum over $\gamma', \gamma'' \in \Gamma$, we obtain (4.8).

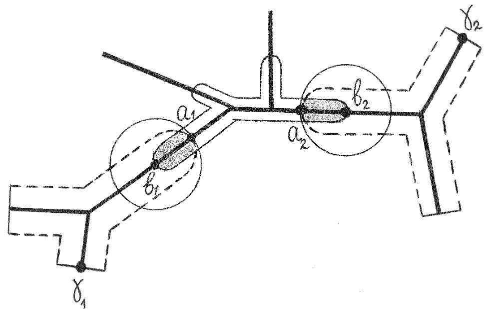

Figure 9. The paths $\pi[b_1, b_2]$ and $\pi[\gamma_1, \gamma_2]$.

(ii) Let Ω be a tree, so that any two of its points are connected through a unique path.

Let $b_1, b_2 \in \operatorname{supp} h$ be a pair of 'extreme points' of $\operatorname{supp} h$ (in figure 9 $\operatorname{supp} h$ is contoured with a solid line), so that $\tau(b_1, b_2) = d[h]$. As is easy to see, on the tree Ω one can choose $\gamma_1, \gamma_2 \in \Gamma$ such that $\pi[\gamma_1, \gamma_2] \supset \pi[b_1, b_2]$.

By this choice, the equalities

$$\tau(\gamma_i, b_i) = \tau(\gamma_i, \operatorname{supp} h),$$

are valid that yields

$$\begin{aligned}\tau(\gamma_1, \gamma_2) &= \tau(\gamma_1, b_1) + \tau(b_1, b_2) + \tau(b_2, \gamma_2)\\ &= \tau(\gamma_1, \operatorname{supp} h) + d[h] + \tau(\gamma_2, \operatorname{supp} h).\end{aligned} \tag{4.11}$$

Fix a small positive ε and take $y_i \in \mathcal{H}_0^1$ such that $\operatorname{supp} y_i \subset B_\varepsilon[b_i]$ and $(h, y_i)_{\mathcal{H}} \neq 0$, $i = 1, 2$ (in figure 9 the balls $B_\varepsilon[b_i]$ are marked with circles). Denote $T_i := \tau(\gamma_i, b_i) + \varepsilon$; following lemma 4, find controls $f_i \in \left(\mathcal{F}_{\gamma_i}^{T_i}\right)_0^1$, $T_i = \tau(\gamma_i, b_i) + \varepsilon$ such that the waves $u^{f_i}(\cdot, T_i)$ are supported in the balls $\overline{B_{T_i}[\gamma_i]}$ (contoured with dotted lines) and satisfy

$$u^{f_i}(\cdot, T_i)|_{e_{a_i}} = y_i,$$

where the sets $e_{a_i} := \operatorname{supp} h \cap B_{T_i}[\gamma_i]$ (shadowed in figure 9) are of the same meaning as in lemma 4. By the choice of f_i one has

$$(h, u^{f_i}(\cdot, T_i))_{\mathcal{H}} = (h, y_i)_{\mathcal{H}} \neq 0;$$

hence,

$$t_{\gamma_i}(h) \leqslant T_i = \tau(\gamma_i, b_i) + \varepsilon = \tau(\gamma_i, \operatorname{supp} h) + \varepsilon.$$

Comparing this with (4.7), by arbitrariness of ε we get

$$t_{\gamma_i}(h) = \tau(\gamma_i, \operatorname{supp} h). \tag{4.12}$$

Returning to (4.11), one has

$$\begin{aligned}d[h] &= \tau(\gamma_1, \gamma_2) - [\tau(\gamma_1, \operatorname{supp} h) + \tau(\gamma_2, \operatorname{supp} h)]\\ &= \langle \text{see } (3.12), (4.12) \rangle = 2t_{\gamma_1,\gamma_2} - [t_{\gamma_1}(h) + t_{\gamma_2}(h)].\end{aligned}$$

So, if Ω is a tree, the maximum in (4.8) is equal to $d[h]$ and we get (4.9). $\square$

In complete analogy with part (ii) of the proof one can establish the following useful result. It remains valid for any graphs, not only trees.

Proposition 4. *For any* $\xi \in \operatorname{int}\Omega$ *and* $c = \text{const} \neq 0$ *the equality*

$$t_\gamma(c\delta_\xi) = \tau(\gamma, \xi), \tag{4.13}$$

holds.

Note in addition, that (4.9) can be invalid on graphs with cycles and we show an example. Let r, α be the polar coordinates on $\mathbf{R}^2$. Denote $\Omega_1 = \{(r, \alpha) \mid r = 1, 0 \leqslant \alpha < 2\pi\}$, $\Omega_2 = \{(r, \alpha) \mid 1 \leqslant r \leqslant 2, \alpha = 0\}$, $\Omega_3 = \left\{(r, \alpha) \mid 1 \leqslant r \leqslant 2, \alpha = \frac{3\pi}{2}\right\}$; consider the graph $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$ and take $\xi = \left(1, \frac{3\pi}{4}\right)$, $h = \delta_\xi$. In this case we have $d[h] = 0$ whereas the right-hand side of (4.8) is equal to $-\pi$.

So, if there exists a nonzero $h \in \mathcal{H}^{-1}$ such that

$$\max_{\gamma', \gamma'' \in \Gamma} \{2t_{\gamma'\gamma''} - [t_{\gamma'}(h) + t_{\gamma''}(h)]\} < 0,$$

then we certainly deal with a graph containing cycles.

5. Inverse problem

5.1. Statement

The boundary spectral inverse problem is to recover an equipped graph (a pair Ω, ρ) from its spectral data. This statement goes back to the classical works by I M Gelfand and B M Levitan (on the Sturm–Liouville operator), M G Krein (inhomogeneous string) and Yu M Berezanskii (multidimensional version). However, a direct analogue of the classical statement would be to recover ρ on a *given* Ω (see [12, 14, 17]). A peculiarity of our version is that one needs to recover an *unknown* graph together with an *unknown* density supported on it.

So, we are going to extract a 'material object' (equipped tree) from the spectral data which are just a set of ordered numbers. The rest of the paper is devoted to a procedure constructing such an object. The role of a 'raw material' is played by Dirac measures or, more precisely, by the set of their spectral images. As we will see, it is the set which may be identified through the spectral data in a natural way.

This approach is traditional for hyperbolic inverse problems solved by the BC method. A noteworthy fact is that recently a similar role of Dirac measures was recognized in an elliptic problem [7]. In all, recalling what has been said in section 4.1 and referring to original papers (see, e.g., [3]), the BC method might be positioned as a trend of thoughts following the slogan 'Extract delta-functions from inverse data!'.

5.2. Spectral representation

Recall that $\{\lambda_k\}_{k=1}^\infty$ and $\{\varphi_k\}_{k=1}^\infty$ are the spectrum and the (orthonormalized) eigenbase of the operator L in $\mathcal{H}$ (see also (2.4)).

Let us introduce a unitary operator (Fourier transform) $U : \mathcal{H} \mapsto \tilde{\mathcal{H}} := l_2$,

$$Uy = \{(y, \varphi_k)_{\mathcal{H}}\}_{k=1}^\infty =: \tilde{y};$$

we call $\tilde{y}$ the *spectral image* of y.

The operator U maps the Sobolev class $\mathcal{H}_0^1$ onto the set

$$\tilde{\mathcal{H}}_0^1 := U\mathcal{H}_0^1 = \left\{ \{c_k\}_{k=1}^\infty \in l_2 \,\middle|\, \sum_k \lambda_k c_k^2 < \infty \right\} \tag{5.1}$$

(summation over $k = 1, 2, \ldots$), the relation

$$(y, w)_{\mathcal{H}_0^1} = \sum_k \lambda_k (y, \varphi_k)_{\mathcal{H}} (w, \varphi_k)_{\mathcal{H}} =: (\tilde{y}, \tilde{w})_{\tilde{\mathcal{H}}_0^1},$$

being valid. Correspondingly, the negative space $\tilde{\mathcal{H}}^{-1}$ of the triple $\tilde{\mathcal{H}}^1_0 \subset \tilde{\mathcal{H}} \subset \tilde{\mathcal{H}}^{-1}$ is

$$\tilde{\mathcal{H}}^{-1} := U\mathcal{H}^{-1} = \left\{ \{c_k\}_{k=1}^{\infty} \,\middle|\, \sum_k \frac{c_k^2}{\lambda_k} < \infty \right\} \tag{5.2}$$

(here we extend U on $\mathcal{H}^{-1}$ defining Uh by

$$(Uh, q)_{\tilde{\mathcal{H}}} := (h, U^{-1}q)_{\mathcal{H}},$$

for $q \in \tilde{\mathcal{H}}^1_0$). Note that the classes $\tilde{\mathcal{H}}^1_0$ and $\tilde{\mathcal{H}}^{-1}$ are determined by the spectrum of L and, thus, by the spectral data.

Transform U maps Dom L onto the set

$$\operatorname{Dom} \tilde{L} = U \operatorname{Dom} L = \left\{ \{c_k\}_{k=1}^{\infty} \in l_2 \,\middle|\, \sum_k \lambda_k^2 c_k^2 < \infty \right\},$$

where

$$\tilde{L} := ULU^{-1} = \operatorname{diag}\{\lambda_k\}_{k=1}^{\infty}.$$

Recall that $1(\cdot) \equiv 1$ on Ω; representing

$$1(\cdot) = \sum_k c_k \varphi_k(\cdot),$$

and calculating the coefficients

$$c_k = (1, \varphi_k)_{\mathcal{H}} = \frac{1}{\lambda_k}(1, L\varphi_k)_{\mathcal{H}} = \langle \text{see (2.3)} \rangle$$

$$= -\frac{1}{\lambda_k} \sum_{e \in E} \int_e \frac{\mathrm{d}^2\varphi_k}{\mathrm{d}^2 e} |\mathrm{d}x| = \langle \text{see (2.2)} \rangle = \frac{1}{\lambda_k} \sum_{\gamma \in \Gamma} \frac{\mathrm{d}\varphi_k}{\mathrm{d}e}(\gamma)$$

one obtains

$$\tilde{1} = U1 = \left\{ \frac{1}{\lambda_k} \sum_{\gamma \in \Gamma} \frac{\mathrm{d}\varphi_k}{\mathrm{d}e}(\gamma) \right\}_{k=1}^{\infty}. \tag{5.3}$$

Let us return to the dynamical system (3.1)–(3.3). Representing

$$u^f(\cdot, t) = \sum_k c_k^f(t) \varphi_k(\cdot),$$

and integrating by parts one can find $c_k^f(t)$ and obtain the spectral image of a wave:

$$\tilde{u}^f(t) = Uu^f(\cdot, t) = \left\{ \sum_{\gamma \in \Gamma} \frac{\mathrm{d}\varphi_k}{\mathrm{d}e}(\gamma) \int_0^t \frac{\sin\sqrt{\lambda_k}(t-s)}{\sqrt{\lambda_k}} f(\gamma, s)\, \mathrm{d}s \right\}_{k=1}^{\infty} \tag{5.4}$$

(see, e.g., [2]). A key fact is that for a given f this image is determined by the spectral data. It enables one to invoke the waves u^f for solving the inverse problem.

Fix $\gamma \in \Gamma$ and recall that the propagator corresponding to γ is $p_\gamma = u^{\delta_\gamma \theta}$. Substituting $f = \delta_\gamma \theta$ in (5.4) we obtain

$$\tilde{p}_\gamma(t) = Up_\gamma(\cdot, t) = \left\{ \frac{\mathrm{d}\varphi_k}{\mathrm{d}e}(\gamma) \frac{1 - \cos\sqrt{\lambda_k} t}{\lambda_k} \right\}_{k=1}^{\infty}. \tag{5.5}$$

Return to the system (3.13)–(3.15) defining the propagator p_ξ and recall that $c_\xi = \|\delta_\xi\|^{-1}_{\mathcal{H}^{-1}}$. The Fourier method easily gives

$$p_\xi(\cdot\,,t) = \sum_k \frac{\sin\sqrt{\lambda_k}t}{\sqrt{\lambda_k}}(c_\xi\delta_\xi, \varphi_k)_{\mathcal{H}}\varphi_k(\cdot),$$

so that

$$\tilde{p}_\xi(t) = Up_\xi(\cdot\,,t) = \left\{\frac{\sin\sqrt{\lambda_k}t}{\sqrt{\lambda_k}}g_k\right\}_{k=1}^{\infty}, \tag{5.6}$$

where $g_k = (c_\xi\delta_\xi, \varphi_k)_{\mathcal{H}}$.

5.3. *Interaction times from spectral data*

Fix a pair of boundary points $\gamma', \gamma'' \in \Gamma$. Calculating the inner product of the corresponding propagators by isometry, one has

$$\begin{aligned}(p_{\gamma'}(\cdot\,,t), p_{\gamma''}(\cdot\,,t))_{\mathcal{H}} &= (\tilde{p}_{\gamma'}(t), \tilde{p}_{\gamma''}(t))_{\tilde{\mathcal{H}}} = \langle \text{see (5.5)} \rangle \\ &= \sum_k \frac{\mathrm{d}\varphi_k}{\mathrm{d}e}(\gamma')\frac{\mathrm{d}\varphi_k}{\mathrm{d}e}(\gamma'')\left[\frac{1-\cos\sqrt{\lambda_k}t}{\lambda_k}\right]^2,\end{aligned}$$

and we gain a possibility of determining the interaction time $t_{\gamma'\gamma''}$ (see section 3.5) from the spectral data:

$$t_{\gamma'\gamma''} = \inf\left\{t > 0 \,\middle|\, \sum_k \frac{\mathrm{d}\varphi_k}{\mathrm{d}e}(\gamma')\frac{\mathrm{d}\varphi_k}{\mathrm{d}e}(\gamma'')\left[\frac{1-\cos\sqrt{\lambda_k}t}{\lambda_k}\right]^2 \neq 0\right\}. \tag{5.7}$$

In addition, due to (3.12) the optical distance $\tau(\gamma', \gamma'') = 2t_{\gamma'\gamma''}$ also turns out to be determined.

For any points $\xi', \xi'' \in \operatorname{int}\Omega$ the interaction time of the corresponding propagators $p_{\xi'}$ and $p_{\xi''}$ may be expressed in terms of the spectral data: by virtue of (5.6) one has

$$t_{\xi'\xi''} = \inf\left\{t > 0 \,\middle|\, \sum_k \left(\frac{\sin\sqrt{\lambda_k}t}{\sqrt{\lambda_k}}\right)^2 g'_k g''_k \neq 0\right\} = \langle \text{see (3.19)} \rangle = \frac{\tau(\xi', \xi'')}{2}, \tag{5.8}$$

where g'_k and g''_k are the Fourier coefficients of $c_{\xi'}\delta_{\xi'}$ and $c_{\xi''}\delta_{\xi''}$.

Recall that the time $t_\gamma(h)$ has been defined in the beginning of section 4.3. To determine it in spectral terms let us note that any $f \in \left(\mathcal{F}^T_\gamma\right)^1_0$ has the form

$$f(\gamma', t) = \delta_\gamma(\gamma')\chi(t),$$

with a $\chi \in H^1_0[0, T]$. Hence, for such an f and an $h \in \mathcal{H}^{-1}$ we have by isometry:

$$(h, u^f(\cdot\,,T))_{\mathcal{H}} = \langle \text{see (5.4)} \rangle = \sum_k h_k \frac{\mathrm{d}\varphi_k}{\mathrm{d}e}(\gamma)\int_0^T \frac{\sin\sqrt{\lambda_k}(T-s)}{\sqrt{\lambda_k}}\chi(s)\,\mathrm{d}s$$

where $h_k = (h, \varphi_k)_{\mathcal{H}}$ are the Fourier coefficients of h. Therefore, one can represent the interaction time as follows:

$$t_\gamma(h) = \inf\left\{T > 0 \,\middle|\, \exists\chi \in H^1_0[0, T] : \sum_k h_k \frac{\mathrm{d}\varphi_k}{\mathrm{d}e}(\gamma)\int_0^T \frac{\sin\sqrt{\lambda_k}(T-s)}{\sqrt{\lambda_k}}\chi(s)\,\mathrm{d}s \neq 0\right\}. \tag{5.9}$$

5.4. Spectral model

In section 2 the optical model of an equipped tree has been introduced. Here we show that an external observer possessing the spectral data can construct one more model (isometric copy) of a tree. More precisely, using the spectral data, we determine a set $\tilde\Omega$, endow it with a metric $\tilde\tau$ and introduce a function $\tilde\rho$ on $\tilde\Omega$ so that the metric space $(\tilde\Omega, \tilde\tau)$ turns out to be isometric to (Ω, τ) whereas $\tilde\rho$ is the pull-back of density ρ through this isometry. Let us realize this plan.

Set $\tilde{\mathcal{D}}_1$. As was declared in section 5.1, the role of a 'material' for constructing the model is played by the set

$$\tilde{\mathcal{D}}_1 = U\mathcal{D}_1 \subset \tilde{\mathcal{H}}^{-1},$$

of spectral images of the normalized Dirac measures. Our nearest goal is to show that an external observer possessing the spectral data is able to select $\tilde{\mathcal{D}}_1$ in $\tilde{\mathcal{H}}^{-1}$.

Introduce a function $\tilde t_\gamma : \tilde{\mathcal{H}}^{-1} \mapsto \bar{\mathbf{R}}_+$,

$$\tilde t_\gamma(g) := t_\gamma(U^{-1}g).$$

In accordance with (5.9), for $g = \{g_k\}_{k=1}^\infty \in \tilde{\mathcal{H}}^{-1}$ one has a representation

$$\tilde t_\gamma(g) = \inf\left\{T > 0 \,\middle|\, \exists\chi \in H_0^1[0, T] : \sum_k g_k \frac{\mathrm{d}\varphi_k}{\mathrm{d}e}(\gamma) \int_0^T \frac{\sin\sqrt{\lambda_k}(T-s)}{\sqrt{\lambda_k}} \chi(s)\,\mathrm{d}s \neq 0\right\}, \tag{5.10}$$

which shows that $\tilde t_\gamma$ is determined by the spectral data.

Introduce a function $\tilde d : \tilde{\mathcal{H}}^{-1} \mapsto \bar{\mathbf{R}}_+$,

$$\tilde d[g] := d[U^{-1}g].$$

By virtue of (4.9) one can represent

$$\tilde d[g] = \max_{\gamma',\gamma''\in\Gamma} \{2t_{\gamma'\gamma''} - [\tilde t_{\gamma'}(g) + \tilde t_{\gamma''}(g)]\}, \tag{5.11}$$

whereas (5.7) and (5.9) enable us to calculate the right-hand side through $\left\{\lambda_k; \frac{\mathrm{d}\varphi_k}{\mathrm{d}e}\big|_\Gamma\right\}_{k=1}^\infty$.

If the function $\tilde d$ is at our disposal, we can characterize the set $\tilde{\mathcal{D}}_1$ as follows: in accordance with (2.5) one has

$$\tilde{\mathcal{D}}_1 = \{g = \{g_k\}_{k=1}^\infty \in \tilde{\mathcal{H}}^{-1} \,|\, \|g\|_{\tilde{\mathcal{H}}^{-1}} = 1, g_1 > 0, \tilde d[g] = 0\}. \tag{5.12}$$

In other words, $\tilde{\mathcal{D}}_1$ coincides with the set of zeros of the function $\tilde d$ lying at a 'hemisphere' $\{g \in \tilde{\mathcal{H}}^{-1} \mid \|g\|_{\tilde{\mathcal{H}}^{-1}} = 1, (g, \tilde\varphi_1)_{\tilde{\mathcal{H}}^{-1}} > 0\}$. Thus, the set $\tilde{\mathcal{D}}_1$ is determined by the spectral data.

Space $(\tilde\Omega, \tilde\tau)$. Let us endow $\tilde{\mathcal{D}}_1$ with a metric. Note that the map $\xi : \tilde{\mathcal{D}}_1 \mapsto \operatorname{int}\Omega$,

$$\xi(g) := \operatorname{supp} U^{-1}g,$$

is a bijection such that $\xi(U(c_\eta\delta_\eta)) = \eta$ for $\eta \in \operatorname{int}\Omega$.

Introduce a function $\tilde\tau : \tilde{\mathcal{D}}_1 \times \tilde{\mathcal{D}}_1 \mapsto \bar{\mathbf{R}}_+$,

$$\tilde\tau(g', g'') := \tau(\xi(g'), \xi(g'')).$$

From this definition it follows at once that the metric spaces $(\tilde{\mathcal{D}}_1, \tilde\tau)$ and $(\operatorname{int}\Omega, \tau)$ are isometric through the map ξ. In accordance with (5.8), for $g' = \{g'_k\}_{k=1}^\infty$, $g'' = \{g''_k\}_{k=1}^\infty$ one has

$$\tilde\tau(g', g'') = 2\inf\left\{t > 0 \,\middle|\, \sum_k \left(\frac{\sin\sqrt{\lambda_k}t}{\sqrt{\lambda_k}}\right)^2 g'_k g''_k \neq 0\right\}, \tag{5.13}$$

so that the function $\tilde\tau$ is determined by the spectrum of L.

Finally, completing $\tilde{\mathcal{D}}_1$ with respect to $\tilde{\tau}$ we get a metric compact $(\tilde{\Omega}, \tilde{\tau})$ isometric to (Ω, τ). Extending the map ξ by continuity one gets an isometry $\xi : (\tilde{\Omega}, \tilde{\tau}) \mapsto (\Omega, \tau)$.

Note in addition that this way of introducing a metric determined by spectral data, in fact, repeats a trick used in [4].

Function $\tilde{\rho}$. Define a function $\tilde{\rho} := \rho \circ \xi$ on $\tilde{\Omega}$ and show that it can be recovered through the spectral data.

For $g = \{g_k\}_{k=1}^{\infty} \in \tilde{\mathcal{D}}_1$ one has

$$\begin{aligned}(p_{\xi(g)}(\cdot\,, t), 1)_{\mathcal{H}} &= (\tilde{p}_{\xi(g)}(t), \tilde{1})_{\tilde{\mathcal{H}}} = \langle \text{see (5.3), (5.6)} \rangle \\ &= \sum_k \left(\frac{\sin\sqrt{\lambda_k} t}{\sqrt{\lambda_k}} g_k \right) \left(\frac{1}{\lambda_k} \sum_{\gamma \in \Gamma} \frac{\mathrm{d}\varphi_k}{\mathrm{d}e}(\gamma) \right),\end{aligned}$$

and

$$\|p_{\xi(g)}(\cdot\,, t)\|^2_{\mathcal{H}} = \|(\tilde{p}_{\xi(g)}(t)\|^2_{\tilde{\mathcal{H}}} = \langle \text{see (5.6)} \rangle = \sum_k \left(\frac{\sin\sqrt{\lambda_k} t}{\sqrt{\lambda_k}} g_k \right)^2 .$$

Thereafter, we can represent

$$\tilde{\rho}(g) = \rho(\xi(g)) = \langle \text{see (3.18)} \rangle = \left\{ \lim_{t \to 0} \frac{\left[\sum_k \left(\frac{\sin\sqrt{\lambda_k} t}{\sqrt{\lambda_k}} g_k \right) \left(\frac{1}{\lambda_k} \sum_\gamma \frac{\mathrm{d}\varphi_k}{\mathrm{d}e}(\gamma) \right) \right]^2}{2t \sum_k \left(\frac{\sin\sqrt{\lambda_k} t}{\sqrt{\lambda_k}} g_k \right)^2} \right\}^2 . \tag{5.14}$$

So, the spectral data determine the function $\tilde{\rho}$ on $\tilde{\mathcal{D}}_1$. Extending by continuity we get $\tilde{\rho}$ on $\tilde{\Omega}$.

A triple $\tilde{\Omega}, \tilde{\tau}, \tilde{\rho}$ is what we call the *spectral model* of the original Ω, τ, ρ

Optical coordinates on $\tilde{\Omega}$. One more ingredient of a spectral model is an intrinsic coordinate system.

As was mentioned in section 2.3 (after lemma 1), the functions $\{\tau(\gamma_j, \cdot)\}_{j=1}^n$ form a coordinate system on Ω. Correspondingly, the family $\{\tau(\gamma_j, \xi(\cdot))\}_{j=1}^n$ is a coordinate system on $\tilde{\Omega}$ which we also call optical.

Optical coordinates may be recovered on $\tilde{\Omega}$ from the spectral data. Indeed, taking $g = U(c_\eta \delta_\eta) \in \tilde{\mathcal{D}}_1$ one has

$$\tau(\gamma_j, \xi(g)) = \tau(\gamma_j, \eta) = \langle \text{see (4.13)} \rangle = t_{\gamma_j}(c_\eta \delta_\eta) = t_{\gamma_j}(U^{-1} g) = \tilde{t}_{\gamma_j}(g),$$

which yields

$$\tau(\gamma_j, \xi(\cdot)) = \tilde{t}_{\gamma_j}, \qquad j = 1, \ldots, n. \tag{5.15}$$

Hence, the functions $\{\tilde{t}_{\gamma_j}(\cdot)\}_{j=1}^n$ can be used as the optical coordinates on $\tilde{\Omega}$ whereas their values can be found by (5.10). **See endnote 3**

From spectral model to optical model. The map $\tilde{i} : \tilde{\Omega} \mapsto \mathbf{R}^n$,

$$\begin{aligned}\tilde{i}(g) &:= i(\xi(g)) = \mathrm{col}\{\tau(\gamma_1, \xi(g)), \ldots, \tau(\gamma_n, \xi(g))\} \\ &= \langle \text{see (5.15)} \rangle = \mathrm{col}\left\{\tilde{t}_{\gamma_1}(g), \ldots, \tilde{t}_{\gamma_n}(g)\right\},\end{aligned} \tag{5.16}$$

associated with the optical coordinates is a bijection onto the image

$$\tilde{i}(\tilde{\Omega}) = i(\xi(\tilde{\Omega})) = i(\Omega) = \Omega_{\text{opt}}.$$

Representing

$$\rho_{\text{opt}} = \rho \circ i^{-1} = (\tilde{\rho} \circ \xi) \circ (\tilde{i} \circ \xi)^{-1} = \tilde{\rho} \circ \tilde{i}^{-1}, \tag{5.17}$$

we see that the map $\tilde{i}$ recovers $\Omega_{\text{opt}}, \tau_{\text{opt}}, \rho_{\text{opt}}$ from $\tilde{\Omega}, \tilde{\tau}, \tilde{\rho}$.

Thus, we arrive at the crucial conclusions:

(i) the spectral data determine the spectral model;

(ii) the spectral model determines the optical model which, in turn, determines the tree up to a spatial isometry.

To solve the inverse problem we need just to summarize these facts.

5.5. Solving the inverse problem

So, we are going to recover a tree Ω and a density ρ on Ω from the spectral data $\left\{\lambda_k;\ \frac{\mathrm{d}\varphi_k}{\mathrm{d}e}\Big|_\Gamma\right\}_{k=1}^\infty$. Recall that by solving the inverse problem we prove theorem 1.

Step 1 (Preliminaries). Set $\tilde{\mathcal{H}} := l_2$ and determine the classes $\tilde{\mathcal{H}}_0^1$ and $\tilde{\mathcal{H}}^{-1}$ by (5.1), (5.2). Find the element $\tilde{1}$ by (5.3). Find the times $t_{\gamma'\gamma''}$ by (5.7). Determine the functions $\tilde{t}_\gamma$ by (5.10) and, then, the function $\tilde{d}$ by (5.11).

Step 2 (Constructing the spectral model). In $\tilde{\mathcal{H}}^{-1}$ select the set $\tilde{\mathcal{D}}_1$ by representation (5.12) and determine the function $\tilde{\tau}$ on $\tilde{\mathcal{D}}_1 \times \tilde{\mathcal{D}}_1$ by (5.13). The pair $(\tilde{\mathcal{D}}_1, \tilde{\tau})$ is a metric space isometric to $(\operatorname{int}\Omega, \tau)$. Completing with respect to metric $\tilde{\tau}$ we get the metric space $(\tilde{\Omega}, \tilde{\tau})$ isometric to (Ω, τ). Then one recovers the function $\tilde{\rho}$ on $\tilde{\Omega}$ by (5.14).

Step 3 (Recovering the optical model). Construct the map $\tilde{i}$ by (5.16) and recover $\Omega_{\text{opt}} = \tilde{i}(\tilde{\Omega})$. Find ρ_{opt} by representation (5.17).

In accordance with lemma 2, the pair $\Omega_{\text{opt}}, \rho_{\text{opt}}$ determines Ω, ρ up to a spatial isometry.

The inverse problem is solved and theorem 1 is proved.

The following is worth noting in addition. In the generic case, the group of spatial isometries of a graph preserving its boundary vertices is trivial. Therefore, if the boundary vertices of the tree *are given* on the plane then, in the generic case, the spectral data determine such a tree uniquely.

6. Applications, open problems, comments

6.1. Wave equation

With the dynamical system (3.1)–(3.3) one associates a (Dirichlet) response operator $R_w^T : \mathcal{F}^T \mapsto \mathcal{F}^T$,

$$R_w^T f := \frac{\mathrm{d}u^f}{\mathrm{d}e}\bigg|_{\Gamma\times[0,T]}.$$

Using the property of controllability of this system [8] and repeating the scheme of [6], one can show that for any fixed $T \geqslant d(\Omega)$ the operator R_w^T determines the (Dirichlet) spectral data uniquely. As a result, for $T \geqslant d(\Omega)$ the operator R_w^T determines Ω, ρ up to a spatial isometry.

6.2. Heat equation

A dynamical system describing heat processes on a graph is

$$\begin{aligned} &u_t + L_{\max} u = 0 && \text{in} \quad \operatorname{int}\Omega \times (0, T),\\ &u|_{t=0} = 0 && \text{in} \quad \Omega,\\ &u = f && \text{on} \quad \Gamma \times [0, T], \end{aligned}$$

function ρ^{-1} has the physical meaning of the heat conductivity, solution $u = u^f(x,t)$ is the temperature. The 'input → output' correspondence is described by a response operator $R_h^T : \mathcal{F}^T \mapsto \mathcal{F}^T$,

$$R_h^T f := \left.\frac{\mathrm{d}u^f}{\mathrm{d}e}\right|_{\Gamma\times[0,T]}.$$

The well-known fact is that for any fixed $T > 0$ the operator R_h^T determines the Dirichlet spectral data. Therefore, for any $T > 0$ the operator R_h^T determines the tree up to a spatial isometry.

6.3. Schrödinger equation

With the system

$$\begin{aligned} &iu_t - L_{\max}u = 0 && \text{in} \quad \operatorname{int}\Omega \times (0,T), \\ &u|_{t=0} = 0 && \text{in} \quad \Omega, \\ &u = f && \text{on} \quad \Gamma \times [0,T], \end{aligned}$$

one associates a response operator $R_s^T : \mathcal{F}^T \mapsto \mathcal{F}^T$,

$$R_s^T f := \left.\frac{\mathrm{d}u^f}{\mathrm{d}e}\right|_{\Gamma\times[0,T]}.$$

Following the scheme of [1], one can prove that for any fixed $T > 0$ the operator R_s^T determines the Dirichlet spectral data uniquely. Hence, for any $T > 0$ the operator R_s^T determines the tree up to a spatial isometry.

6.4. Inverse scattering

Assume that all the edges of a graph Ω incident to the boundary vertices can be extended (as nonintersecting straight lines) to infinity and the density of such a noncompact graph satisfies $\rho \equiv 1$ for $|x| \geqslant R$ with an R large enough. In this case one can associate with the graph the time-domain (see, e.g., [10]) or frequency-domain (see, [12–14]) scattering data.

If Ω is a tree and the scattering data are such that they determine the response operator R_w^T of the unknown compact part $\Omega \cap B_R[0]$ for $T > 2R$, then these data determine the compact part (together with density on it) up to a spatial isometry.

6.5. Another data and operators

It presents no problems to modify our scheme for the Neumann spectral data and the Neumann response operators.

The scheme may be easily adapted for the case of the operators $L = -\frac{\mathrm{d}}{\mathrm{d}e}\rho\frac{\mathrm{d}}{\mathrm{d}e}$ and $L = -\frac{\mathrm{d}^2}{\mathrm{d}e^2} + q$.

6.6. Open problems

- Sure, the first challenging problem is to extend our approach to graphs with cycles. The principal difficulties are the following:

(i) cycles can imply a lack of the global controllability of the graph; as a result, some functionals in $\mathcal{H}^{-1}$ may be invisible from the boundary (see, e.g., [2]);

(ii) the basic formula (4.9) does not hold and, therefore, our trick of selecting the set $\tilde{\mathcal{D}}_1$ in $\tilde{\mathcal{H}}^{-1}$ with the help of the characterization (5.12) does not work.

A noteworthy fact is that, at least in some cases, an external observer is able to check whether the graph is a tree. Indeed, the right-hand side of the general relation (4.8) can be found through the spectral data for *any* graph. If it takes negative values, the graph necessarily contains cycles (see the last paragraph of section 4.3).

- In the case of trees, an important problem is to find an efficient method of determining the zeros of the function $\tilde{d}$ realizable numerically. It is far from being evident that such a method exists. Indeed, the diameter $d[h]$ depends on $h \in \mathcal{H}^{-1}$ not continuously and, as a result, $\tilde{d}[g]$ is discontinuous at *every* $g \in \tilde{\mathcal{H}}^{-1}$. Is so exotic an object available for calculations?
- Another problem is to recover a part of the tree from the response operator R_w^T given for a *prescribed* $T < d(\Omega)$. The case of inverse data given on a part of Γ is also of interest.

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Endnotes

(1) Author: Please check that the sense of the sentence is still correct after changes have been made 'The graph ... on the edges'.
(2) Author: The meaning of the sentence 'Therefore, ... unsolvable' is not clear. Please check.
(3) Author: Please check that the sense of the sentence 'Hence, the functions ... by (5.10)' is still correct after changes have been made.
(4) Author: Please provide page number in reference [4].
(5) Author: Please confirm the journal title whether '*Izv. Vyssh. Uchebn. Zaved. Mat.*' or '*Izv. Uchebn. Zaved. Mat.*' in reference [16].
(6) Author: Figures 1, 2 and 4 are not cited in the text. Please check.