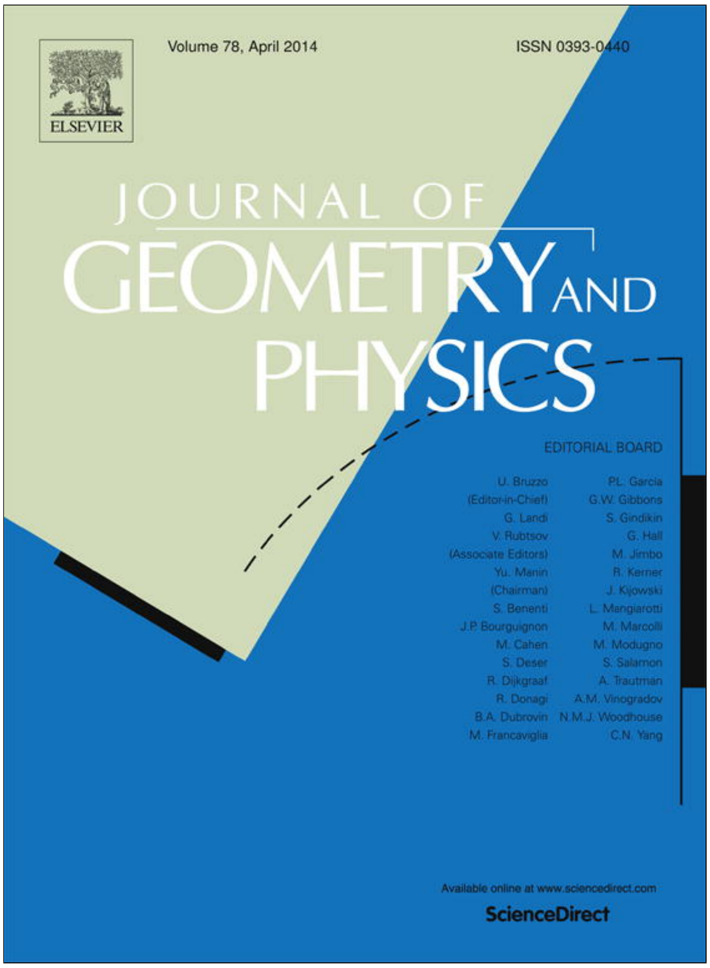
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Elements of noncommutative geometry in inverse problems on manifolds

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ABSTRACT

We deal with two dynamical systems associated with a Riemannian manifold with boundary. The first one is a system governed by the scalar wave equation, and the second is governed by the Maxwell equations. Both systems are controlled from the boundary. The inverse problem is to recover the manifold from measurements on the boundary (inverse data).

We show that the inverse data determine C*-algebras, whose (topologized) spectra are identical to the manifold. For this reason, to recover the manifold one can determine a proper algebra from the inverse data, find its spectrum, and provide the spectrum with a Riemannian structure.

The paper develops an algebraic version of the boundary control method, which is an approach to inverse problems based on their relations to control theory.

1. Introduction

About the paper

One of the basic theses of noncommutative geometry is that a topological space can be characterized in terms of an algebra associated with it (see [1–3]). In other words, a space can be encoded into an algebra. As was recognized in [4,5], such a coding is quite relevant and efficient for solving inverse problems on manifolds. In particular, it enables one to recover a Riemannian manifold from its dynamical or spectral *boundary inverse data*.

What does "to recover a manifold" mean? From the physical viewpoint, the inverse data formalize the measurements that the external observer implements on the boundary. In our case, the role of data is played by the so-called *response operator* R. It describes the reply of the dynamical system associated with the manifold to the action of boundary controls, and the reply is also measured on the boundary.

In inverse problems, the principal question is: To what extent do the inverse data determine the manifold (its topology, metric, etc. [6,7])? In particular, is it possible to reconstruct the manifold from the data?

Having the goal to determine Ω from R, the observer must take into account the obvious nonuniqueness of such a determination. Indeed, let two manifolds Ω and Ω' have a mutual boundary $\partial\Omega = \partial\Omega'$, and let $i : \Omega \to \Omega'$ be an isometry such that $i|_{\partial\Omega} = \mathrm{id}$. In this case, their boundary inverse data turn out to be identical: $R = R'$. Hence, the correspondence $\Omega \mapsto R$ is not injective and to recover the original Ω from R is impossible. In other words, the observer is not able to distinguish Ω from Ω' in principle.[1]

In such a situation, the only reasonable understanding of the reconstruction problem is the following: *Given R, construct a manifold $\tilde{\Omega}$ such that $\tilde{R} = R$.*

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[1] Such an impossibility is of very general character in system theory: see [8, Chap. 10].

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How algebras are used. We show that a Riemannian manifold Ω can be identified with the (topologized) spectrum $\widehat{\mathfrak{A}(\Omega)}$ of an appropriate Banach algebra $\mathfrak{A}(\Omega)$, the algebra being determined by the inverse data up to isometric isomorphism. Therefore, one can reconstruct Ω in accordance with the following plan:

- extract an isometric copy $\tilde{\mathfrak{A}}(\Omega)$ of $\mathfrak{A}(\Omega)$ from R
- find its spectrum $\widehat{\tilde{\mathfrak{A}}(\Omega)} =: \tilde{\Omega}$, which is homeomorphic to $\widehat{\mathfrak{A}(\Omega)}$ by virtue of $\tilde{\mathfrak{A}}(\Omega) \overset{\text{isom}}{=} \mathfrak{A}(\Omega)$. Thus, we have $\tilde{\Omega} \overset{\text{hom}}{=} \Omega$
- endow $\tilde{\Omega}$ with a proper Riemannian structure.

As a result, we get a Riemannian manifold $\tilde{\Omega}$ isometric to the original Ω. It is $\tilde{\Omega}$, which solves the reconstruction problem: we have $\tilde{R} = R$ by construction.

Our paper keeps to this plan and extends it to the inverse problem of electrodynamics.

Contents

We deal with a smooth compact Riemannian manifold Ω with boundary. All the functional spaces and classes, as well as the algebras under consideration, are *real*.

Eikonals. We introduce the *eikonals*, which play the role of the main instrument for reconstruction. An eikonal $\tau_\sigma(\cdot) = \operatorname{dist}(\cdot, \sigma)$ is a distance function on Ω with the base $\sigma \subset \partial\Omega$. As is shown, eikonals determine the Riemannian structure on Ω.

With each eikonal one associates the self-adjoint operator $\check{\tau}_\sigma$ in $L_2(\Omega)$ that multiplies functions by τ_σ. Its representation, by the Spectral Theorem, is $\check{\tau}_\sigma = \int_0^\infty s\, dX_\sigma^s$, where X_σ^s is the projection onto the subspace $L_2(\Omega^s[\sigma])$ of functions supported in the metric neighborhood $\Omega^s[\sigma] \subset \Omega$ of σ of radius s.

For an oriented 3d-manifold Ω, by analogy with the scalar case, we introduce the *solenoidal eikonals*[2] $\varepsilon_\sigma = \int_0^\infty s\, dY_\sigma^s$, which act in the space $\mathcal{C} = \{\operatorname{curl} h \mid h, \operatorname{curl} h \in \vec{L}_2(\Omega)\}$, relevant in electrodynamics. Here Y_σ^s projects vector fields onto the subspace of curls supported in $\Omega^s[\sigma]$.

Algebras. As we show, the eikonals $\{\tau_\sigma \mid \sigma \subset \partial\Omega\}$ generate the Banach algebra $C(\Omega)$ of continuous functions. Its spectrum[3] $\widehat{C(\Omega)}$ is homeomorphic to Ω (see [9,10]).

The operator eikonals $\{\check{\tau}_\sigma \mid \sigma \subset \partial\Omega\}$ generate the operator algebra $\mathfrak{T}$, which is a commutative subalgebra of the bounded operator algebra $\mathfrak{B}(L_2(\Omega))$. The algebras $\mathfrak{T}$ and $C(\Omega)$ are isometrically isomorphic (via $\check{\tau}_\sigma \mapsto \tau_\sigma$). Hence, their spectra are homeomorphic, and we have $\widehat{\mathfrak{T}} \overset{\text{hom}}{=} \widehat{C(\Omega)} \overset{\text{hom}}{=} \Omega$.

The solenoidal eikonals generate the operator algebra $\mathfrak{E}$, which is a subalgebra of $\mathfrak{B}(\mathcal{C})$. In contrast to $\mathfrak{T}$, the algebra $\mathfrak{E}$ is *noncommutative*. However, the factor-algebra $\dot{\mathfrak{E}} = \mathfrak{E}/\mathfrak{K}$ over the ideal of compact operators $\mathfrak{K} \subset \mathfrak{E}$ turns out to be commutative. Moreover, one has $\dot{\mathfrak{E}} \overset{\text{isom}}{=} C(\Omega)$, which implies $\widehat{\dot{\mathfrak{E}}} \overset{\text{hom}}{=} \widehat{C(\Omega)} \overset{\text{hom}}{=} \Omega$.

Inverse problems. Following [5], we begin with a dynamical system, which is governed by the scalar *wave equation* in Ω and controlled from the boundary $\partial\Omega$. The input$\mapsto$output correspondence is realized by a *response operator R*, which plays the role of inverse data. A reconstruction (inverse) problem is to recover the manifold Ω from the given R.

Solving this problem, we construct (via R) an operator algebra $\tilde{\mathfrak{T}}$ isometric to $\mathfrak{T}$, find its spectrum $\tilde{\Omega} := \widehat{\tilde{\mathfrak{T}}} \overset{\text{hom}}{=} \widehat{\mathfrak{T}} \overset{\text{hom}}{=} \Omega$, endow it with the Riemannian structure, using the images of eikonals, and eventually convert $\tilde{\Omega}$ into an isometric copy of the original manifold Ω. The copy $\tilde{\Omega}$ provides the solution to the reconstruction problem.

In electrodynamics, the corresponding system is governed by the *Maxwell equations* and also controlled from the boundary. The relevant response operator R plays the role of inverse data for the reconstruction problem. To solve this problem, we repeat all the steps of the above-described procedure. The only additional step is the factorization $\mathfrak{E} \mapsto \dot{\mathfrak{E}}$, which eliminates noncommutativity.

Comments

Reconstruction via algebras is known in Noncommutative Geometry: see [1–3]. However, there is a substantial difference: in the mentioned papers the starting point for reconstruction is the so-called spectral triple $\{\mathcal{A}, \mathcal{H}, \mathcal{D}\}$, which consists of a commutative algebra, a Hilbert space, and a self-adjoint (Dirac-like) operator. So, an algebra *is given*. In our case, we must first *extract* the algebra from R. Then we deal with this algebra imposed by inverse data, whereas its "good" properties are not guaranteed.

Reconstruction via algebras in inverse problems was originated in [4] and developed in [5]. It represents an algebraic version of the *boundary control method*, which is an approach to inverse problems based on their relations to control theory (see [6,7]). We hope for further applications of this version to inverse problems of mathematical physics.

2 "Solenoidal" means *divergence free*.
3 The set of multiplicative functionals topologized by Gelfand.

2. Eikonals

We deal with a smooth[4] connected compact Riemannian manifold Ω with the boundary Γ, g is the metric tensor, $\dim \Omega = n \geqslant 2$.

For a set $A \subset \Omega$, by

$$\Omega^r[A] := \{x \in \Omega \mid \operatorname{dist}(x, A) < r\}, \quad r > 0$$

we denote its metric r-neighborhood. Compactness implies $\operatorname{diam} \Omega := \sup\{\operatorname{dist}(x, y) \mid x, y \in \Omega\} < \infty$ and

$$\Omega^r[A] = \Omega \quad \text{for } r > \operatorname{diam} \Omega. \tag{2.1}$$

2.1. Scalar eikonals

We say that a subset $\sigma \subset \Gamma$ is *regular* and write $\sigma \in \mathcal{R}(\Gamma)$ if σ is diffeomorphic to a "disk" $\{p \in \mathbb{R}^{n-1} \mid \|p\| \leqslant 1\}$.

We call a distance function of the form

$$\tau_\sigma(x) := \operatorname{dist}(x, \sigma), \quad x \in \Omega \, (\sigma \in \mathcal{R}(\Gamma)),$$

a (scalar) *eikonal*. The set σ is said to be a *base*. Eikonals are Lipschitz functions: $\tau_\sigma \in \operatorname{Lip}(\Omega) \subset C(\Omega)$. Moreover, eikonals are smooth almost everywhere and

$$|\nabla \tau_\sigma(x)| = 1 \quad \text{a.a. } x \in \Omega \tag{2.2}$$

holds. Also, mention the following simple geometric facts.

Lemma 1. *For any $x \in \Omega$ there is $\sigma \in \mathcal{R}(\Gamma)$ such that $\tau_\sigma(x) \neq 0$. For any distinct $x, y \in \Omega$ there is a $\sigma \in \mathcal{R}(\Gamma)$ such that $\tau_\sigma(x) \neq \tau_\sigma(y)$ (i.e., the eikonals distinguish points of Ω). The equality $\sigma = \{\gamma \in \Gamma \mid \tau_\sigma(\gamma) = 0\}$ holds.*

Proof. The first and third assertions are obvious. Consider the second one. Suppose that for every $\sigma \in \mathcal{R}(\Gamma)$ we have

$$\tau_\sigma(x) = \tau_\sigma(y)$$

for some $x, y \in \Omega$. Let γ_x and γ_y be the points of the boundary Γ (may be non-unique) nearest to x and y respectively. Since the last equality holds true for arbitrarily small sets $\sigma \in \mathcal{R}(\Gamma)$ containing γ_x (or γ_y), we have

$$\operatorname{dist}(x, \gamma_x) = \operatorname{dist}(y, \gamma_x), \qquad \operatorname{dist}(x, \gamma_y) = \operatorname{dist}(y, \gamma_y).$$

The inequality $\operatorname{dist}(y, \gamma_x) \geq \operatorname{dist}(y, \gamma_y)$ implies

$$\operatorname{dist}(x, \gamma_x) \geq \operatorname{dist}(x, \gamma_y),$$

from which, and with regard to $\operatorname{dist}(x, \gamma_x) \leq \operatorname{dist}(x, \gamma_y)$, we obtain

$$\operatorname{dist}(x, \gamma_x) = \operatorname{dist}(x, \gamma_y).$$

Hence, $\gamma_y \in \Gamma$ is also the nearest point to x. Both geodesics connecting γ_y with x and y are orthogonal to Γ and (by the last equality) have the same length. Therefore, these geodesics coincide, and we arrive at $x = y$. □

Copy $\tilde{\Omega}$

As functions on Ω, eikonals are determined by the Riemannian structure of Ω. The converse is also true in the following sense.

Assume that we are given a topological space $\tilde{\Omega}$, which is homeomorphic to Ω (with the Riemann metric topology) via a homeomorphism $\eta : \Omega \to \tilde{\Omega}$; let $\tilde{\tau}_\sigma := \tau_\sigma \circ \eta^{-1}$. Also, assume that η *is unknown* but we are given the map

$$\mathcal{R}(\Gamma) \ni \sigma \mapsto \tilde{\tau}_\sigma \in C(\tilde{\Omega}). \tag{2.3}$$

Then one can endow $\tilde{\Omega}$ with the Riemannian structure, which converts it into a manifold *isometric* to Ω. Roughly speaking, the way is the following (see [11] for details).

For a fixed point $p \in \tilde{\Omega}$ one can find its neighborhood $\omega \subset \tilde{\Omega}$ and sets $\sigma_1, \ldots, \sigma_n \in \mathcal{R}(\Gamma)$ such that the functions $x^1 = \tilde{\tau}_{\sigma_1}(\cdot), \ldots, x^n = \tilde{\tau}_{\sigma_n}(\cdot)$ constitute a coordinate chart $\phi : \omega \ni p \mapsto \{x^k(p)\}_{k=1}^n \in \mathbb{R}^n$. The coordinates endow ω with tangent spaces. These spaces can be provided with the metric tensor $\tilde{g} = \eta_* g$: one can determine its components $\tilde{g}^{ij}$ from the equations

$$\tilde{g}^{ij}(x) \, \frac{\partial \tilde{\tau}_\sigma \circ \phi^{-1}}{\partial x^i}(x) \, \frac{\partial \tilde{\tau}_\sigma \circ \phi^{-1}}{\partial x^j}(x) = 1, \quad x \in \phi(\omega), \; \sigma \in \mathcal{R}(\Gamma) \tag{2.4}$$

which are just (2.2) written in coordinates. Choosing here $\sigma = \sigma_i$, we get $\tilde{g}^{ii} = 1$. Choosing (a finite number of) additional sets σ, we can determine the functions $\frac{\partial \tilde{\tau}_\sigma \circ \phi^{-1}}{\partial x^i}$ and then find all other components $\tilde{g}^{ij}(x)$ by solving the system (2.4) with respect to them.

[4] Everywhere in the paper, "smooth" means C^∞-smooth.

So, although the homeomorphism η is unknown, we are able to endow $\tilde{\Omega}$ with the metric tensor $\tilde{g} = \eta_* g$, which converts it into a Riemannian manifold $(\tilde{\Omega}, \tilde{g})$ isometric to (Ω, g) by construction.

Moreover, there is a natural way of identifying the boundaries $\tilde{\Gamma} := \partial\tilde{\Omega}$ and $\Gamma = \partial\Omega$. First, we can select the boundary points in $\tilde{\Omega}$ by

$$\tilde{\Gamma} = \bigcup_{\sigma \in \mathcal{R}(\Gamma)} \tilde{\sigma}, \quad \text{where } \tilde{\sigma} := \{\tilde{\gamma} \in \tilde{\Omega} \mid \tilde{\tau}_\sigma(\tilde{\gamma}) = 0\}.$$

Then we identify $\Gamma \ni \gamma \equiv \tilde{\gamma} \in \tilde{\Gamma}$ if $\gamma \in \sigma$ implies $\tilde{\gamma} \in \tilde{\sigma}$ for all regular σ containing γ.

As a result, we get the manifold $(\tilde{\Omega}, \tilde{g})$ isometric to (Ω, g), these manifolds having the mutual boundary Γ. In what follows we refer to $(\tilde{\Omega}, \tilde{g})$ as a canonical copy of the original manifold Ω (briefly, the *copy* $\tilde{\Omega}$).

The aforesaid is summarized as follows.

Proposition 1. *The space $\tilde{\Omega}$, along with the map (2.3), determines the copy $\tilde{\Omega}$ and, hence, determine Ω up to isometry of Riemannian manifolds.*

2.2. Operator eikonals

Introduce the space $\mathcal{H} := L_2(\Omega)$ with the inner product

$$(u, v)_{\mathcal{H}} = \int_\Omega u(x)v(x)\, dx$$

(dx is the Riemannian volume element). Let $A \subset \Omega$ be a measurable subset, $\chi_A(\,\cdot\,)$ its indicator (a characteristic function). By

$$\mathcal{H}\langle A\rangle := \{\chi_A y \mid y \in \mathcal{H}\}$$

we denote the subspace of functions supported on A. The (orthogonal) projection X_A in $\mathcal{H}$ onto $\mathcal{H}\langle A\rangle$ multiplies functions by χ_A, i.e., cuts off functions on A.

Let $\mathfrak{B}(\mathcal{H})$ be the normed algebra of bounded operators in $\mathcal{H}$. With a scalar eikonal τ_σ one associates an operator $\check{\tau}_\sigma \in \mathfrak{B}(\mathcal{H})$, which acts in $\mathcal{H}$ by the rule

$$\left(\check{\tau}_\sigma y\right)(x) := \tau_\sigma(x)\, y(x), \quad x \in \Omega \tag{2.5}$$

and is bounded since Ω is compact. Moreover, one has

$$\|\check{\tau}_\sigma\| = \max_{x\in\Omega} \tau_\sigma(x) = \|\tau_\sigma\|_{C(\Omega)} \leqslant \operatorname{diam} \Omega. \tag{2.6}$$

With a slight abuse of terms, we also call $\check{\tau}_\sigma$ an *eikonal*.

Each eikonal is a self-adjoint positive operator, which is represented by the Spectral Theorem in a well-known form.

Proposition 2. *The representation*

$$\check{\tau}_\sigma = \int_0^\infty s\, dX_\sigma^s \tag{2.7}$$

is valid, where the projections $X_\sigma^s := X_{\Omega^s[\sigma]}$ cut off functions on the metric neighborhoods of σ.

Note that the integration interval is finite since for $s > \max_{x\in\Omega} \tau_\sigma(x)$ the projection X_σ^s is equal to the identity operator.

The eikonals corresponding to different bases do commute. This follows from the commutation of X_σ^s and $X_{\sigma'}^{s'}$ for all $\sigma, \sigma' \in \mathcal{R}(\Gamma)$ and $s, s' \geqslant 0$.

2.3. Solenoidal operator eikonals

Here we introduce an analog of $\check{\tau}_\sigma$ related to electrodynamics.

3d-manifold

Now, let $\dim \Omega = 3$. Also, let Ω be orientable, g the metric tensor, μ the Riemannian volume 3-form. On such a manifold, the intrinsic operations of vector analysis are well defined on smooth functions and vector fields (sections of the tangent bundle $T\Omega$). Recall their definitions (see, e.g., [12]).

- For a field a, one defines the *conjugate* 1-*form* $a_\sharp$ by $a_\sharp(b) = g(a, b)$ for any field b. For a 1-form ω, the *conjugate field* $\omega^\sharp$ is defined by $g(\omega^\sharp, b) = \omega(b)$ for any field b.
- *scalar product* $\cdot\ :\ \{\text{fields}\} \times \{\text{fields}\} \to \{\text{functions}\}$ is defined pointwise by $a \cdot b = g(a, b)$. The *vector product* $\wedge\ :\ \{\text{fields}\} \times \{\text{fields}\} \to \{\text{fields}\}$ is defined pointwise by $g(a \wedge b, c) = \mu\,(a, b, c)$ for any field c.

- The *gradient* ∇ : {functions} $\to$ {fields} and *divergence* div : {fields} $\to$ {functions} are defined by $\nabla\varphi := (d\varphi)^\sharp$ and $\operatorname{div} a := \star d \star a_\sharp$, respectively, where d is the exterior derivative and $\star$ is the Hodge operator.
- The *curl* is a map curl : {fields} $\to$ {fields}, $\operatorname{curl} a := (\star d\, a_\sharp)^\sharp$. Recall the basic identities div curl $= 0$ and curl $\nabla = 0$.
- The *Laplacian* Δ : {functions} $\to$ {functions} is $\Delta := \operatorname{div} \nabla$.

Note that these operations can also be understood in the sense of distributions. We will use the following formulas of vector analysis:

$$\operatorname{div}(\varphi u) = \nabla\varphi \cdot u + \varphi \operatorname{div} u, \tag{2.8}$$

$$\operatorname{div}(u \wedge v) = \operatorname{curl} u \cdot v - u \cdot \operatorname{curl} v, \tag{2.9}$$

$$\operatorname{curl}(\varphi u) = \nabla\varphi \wedge u + \varphi \operatorname{curl} u. \tag{2.10}$$

In (2.8) and (2.10), the function φ is Lipschitz; the field u is locally integrable and its divergence is also locally integrable. In (2.9) we may suppose that u or v is Lipschitz, and the other field is locally integrable and has locally integrable curl .

By $\nu = \nu(\gamma),\ \gamma \in \Gamma$ we denote the unit outward normal to the boundary. Let $y \in T\Omega$ be a smooth field *tangent* on Γ, i.e., $\nu \cdot y = 0$ everywhere on Γ. Its trace $y|_\Gamma$ is canonically identified with the proper element of $T\Gamma$ (see [12]), and we regard $y|_\Gamma \in \vec{C}^\infty(\Gamma)$.

By $\vec{L}_2(\Gamma)$ we denote the space of square integrable fields on Γ with the inner product

$$(a, b)_{\vec{L}_2(\Gamma)} := \int_\Gamma a(\gamma) \cdot b(\gamma)\, d\gamma,$$

where $d\gamma$ is the canonical surface element on Γ.

Solenoidal spaces

The class of smooth fields $\vec{C}^\infty(\Omega)$ is dense in the space $\vec{\mathcal{H}} := \vec{L}_2(\Omega)$ with the product

$$(u, v)_{\vec{\mathcal{H}}} = \int_\Omega u(x) \cdot v(x)\, dx.$$

This space contains the (sub)spaces

$$\mathcal{J} := \{y \in \vec{\mathcal{H}} \mid \operatorname{div} y = 0 \text{ in } \Omega\}, \qquad \mathcal{C} := \{\operatorname{curl} h \in \vec{\mathcal{H}} \mid h, \operatorname{curl} h \in \vec{\mathcal{H}}\} \subset \mathcal{J}$$

of *solenoidal* (i.e., divergence free) fields and curls. Note that the smooth classes $\mathcal{J} \cap \vec{C}^\infty(\Omega)$ and $\mathcal{C} \cap \vec{C}^\infty(\Omega)$ are dense in $\mathcal{J}$ and $\mathcal{C}$, respectively.

Let $H_0^1(\Omega)$ be the Sobolev class of functions vanishing at Γ, ν be the unit outward normal to Γ. Recall the well-known decompositions

$$\vec{\mathcal{H}} = \mathcal{G}_0 \oplus \mathcal{J} = \mathcal{G}_0 \oplus \mathcal{C} \oplus \mathcal{D}, \tag{2.11}$$

where $\mathcal{G}_0 := \{\nabla q \mid q \in H_0^1(\Omega)\}$ is the space of *potential fields*, $\mathcal{D} := \{y \in \mathcal{J} | \operatorname{curl} y = 0,\ \nu \wedge y = 0 \text{ on } \Gamma\}$ is a finite-dimensional subspace of harmonic *Dirichlet fields* [12].

For an $A \subset \Omega$ we denote

$$\vec{\mathcal{H}}\langle A\rangle := \{\chi_A y \mid y \in \vec{\mathcal{H}}\}, \qquad \mathcal{J}\langle A\rangle := \overline{\{y \in \mathcal{J} \mid \operatorname{supp} y \subset A\}},$$

$$\mathcal{C}\langle A\rangle := \overline{\{\operatorname{curl} h \mid h \in \vec{C}^\infty(\Omega),\ \operatorname{supp} h \subset A\}}$$

(the closure in $\vec{\mathcal{H}}$) the subspaces of fields supported in A.

Projections Y_σ^s

Fix a $\sigma \in \mathcal{R}(\Gamma)$ and take $A = \Omega^s[\sigma]$. Let Y_σ^s be the projection in $\mathcal{C}$ onto the subspace $\mathcal{C}\langle\Omega^s[\sigma]\rangle$. In contrast to the projections X_σ^s, the action of Y_σ^s is not reduced to cutting off fields to $\Omega^s[\sigma]$, it acts in a more complicated way.

Namely, let $\Omega^s[\sigma]$ be homeomorphic to a ball in $\mathbb{R}^3$, which holds for s small enough. In this case, any solenoidal field supported in $\Omega^s[\sigma]$ is a curl (has a vector potential), so that $\mathcal{C}\langle\Omega^s[\sigma]\rangle = \mathcal{J}\langle\Omega^s[\sigma]\rangle$ holds. The projection to the latter subspace is of a well-known form (see [7,11]). With the help of it, we get the representation

$$Y_\sigma^s y = \begin{cases} y - \nabla p & \text{in } \Omega^s[\sigma], \\ 0 & \text{in } \Omega \backslash \Omega^s[\sigma], \end{cases} \tag{2.12}$$

where p is the solution to the elliptic Dirichlet–Neumann problem

$$\Delta p = 0 \quad \text{in int}\, \Omega^s[\sigma],$$

$$p = 0 \quad \text{on } \big(\partial\Omega^s[\sigma]\big) \cap \Gamma,$$

$$\nu \cdot \nabla p = \nu \cdot y \quad \text{on } \big(\partial\Omega^s[\sigma]\big) \backslash \Gamma;$$

here "int" is the set of interior points. For large s, the domain $\Omega^s[\sigma]$ may be of more complicated topology. As a result, $\mathcal{J}\langle\Omega^s[\sigma]\rangle \ominus \mathcal{C}\langle\Omega^s[\sigma]\rangle$ may contain harmonic fields and Y_σ^s acts in a more complicated way.

Anyway, representation (2.12) shows that Y^s_σ is not a local operator: the property $\operatorname{supp} Y^s_\sigma y \subset \operatorname{supp} y$ does not hold. As a consequence, in contrast to the cutting off operators X^s_σ, the projections Y^s_σ and $Y^{s'}_{\sigma'}$ *do not commute*. Indeed, in the generic case, $Y^s_\sigma Y^{s'}_{\sigma'} y$ and $Y^{s'}_{\sigma'} Y^s_\sigma y$ are supported in $\Omega^s[\sigma]$ and $\Omega^{s'}[\sigma']$ respectively but not necessarily in $\Omega^s[\sigma] \cap \Omega^{s'}[\sigma']$.

Eikonals ε_σ

By analogy with (2.7), define a *solenoidal operator eikonal*

$$\varepsilon_\sigma := \int_0^\infty s\, dY^s_\sigma, \tag{2.13}$$

which is a self-adjoint operator in $\mathcal{C}$. As in (2.7), the integration interval in (2.13) is finite since for $s > \max_{x\in\Omega} \tau_\sigma(x)$ the projection Y^s_σ is equal to the identity operator in $\mathcal{C}$. Hence, the operator ε_σ is bounded.

An important fact is that, as a consequence of the noncommutativity of Y^s_σ and $Y^{s'}_{\sigma'}$, the eikonals ε_σ and $\varepsilon_{\sigma'}$ also *do not commute*.

Multiplying a field $h \in \mathcal{C}$ by a bounded function φ, one takes the field out of the subspace of curls: $\varphi h \in \vec{\mathcal{H}}$ but $\varphi h \notin \mathcal{C}$ in general. However, a map $h \mapsto \varphi h$ is a well-defined bounded operator from $\mathcal{C}$ to $\vec{\mathcal{H}}$. For instance, interpreting $\check{\tau}_\sigma$ as an operator that multiplies vector fields by the scalar eikonal τ_σ, we have $\check{\tau}_\sigma \in \mathfrak{B}(\mathcal{C}; \vec{\mathcal{H}})$.

The following result is of crucial character for future application to inverse problems. By $\mathfrak{K}(\mathcal{C}; \vec{\mathcal{H}}) \subset \mathfrak{B}(\mathcal{C}; \vec{\mathcal{H}})$ we denote the set of compact operators.

Theorem 1. *For any $\sigma \subset \Gamma$ the relation $\varepsilon_\sigma - \check{\tau}_\sigma \in \mathfrak{K}(\mathcal{C}; \vec{\mathcal{H}})$ holds.*

In the proof (see Section 5.1), the techniques developed in [13] are used.

3. Algebras

3.1. Handbook

We begin with minimal information about algebras: for details, see, e.g., [9,10]. The abbreviations BA and CBA mean a Banach and commutative Banach algebra, respectively.

1. A BA is a (complex or real) Banach space $\mathcal{A}$ equipped with multiplication operation ab satisfying $\|ab\| \le \|a\|\,\|b\|$ $a, b \in \mathcal{A}$. We deal with algebras with the unit $e \in \mathcal{A} : ea = ae = a$.
   A BA $\mathcal{A}$ is said to be commutative if $ab = ba$ for all $a, b \in \mathcal{A}$. *Example*: the algebra $C(X)$ of continuous functions with norm $\|a\| = \sup_X |a(\cdot)|$ on a topological space X. The subalgebras of $C(X)$ are called *function algebras*.
   A CBA is said to be uniform if $\|a^2\| = \|a\|^2$ holds. All function algebras are uniform.
2. Let $\mathcal{A}'$ be the dual space. A nonzero functional $\delta \in \mathcal{A}'$ is called multiplicative if $\delta(ab) = \delta(a)\delta(b)$. *Example*: a Dirac measure $\delta_{x_0} \in C'(X) : \delta_{x_0}(a) = a(x_0)$ $(x_0 \in X)$. The set of multiplicative functionals endowed with $*$-weak topology (in $\mathcal{A}'$) is called a *spectrum* of $\mathcal{A}$ and denoted by $\widehat{\mathcal{A}}$. The spectrum is a compact Hausdorff space.
3. The *Gelfand transform* acts from a CBA $\mathcal{A}$ to $C(\widehat{\mathcal{A}})$ by the rule $G : a \mapsto a(\cdot)$, $a(\delta) := \delta(a)$, $\delta \in \widehat{\mathcal{A}}$. It represents $\mathcal{A}$ as a function algebra. Note that the passage from $\mathcal{A}$ to $G\mathcal{A} \subset C(\widehat{\mathcal{A}})$ is referred to as *geometrization* of $\mathcal{A}$.

   Theorem 2 (*I.M. Gelfand*)**.** *If $\mathcal{A}$ is a uniform CBA, then G is an isometric isomorphism from $\mathcal{A}$ onto $G\mathcal{A}$, i.e., $G(\alpha a+\beta b+cd) = \alpha Ga + \beta Gb + Gc\, Gd$ and $\|Ga\|_{C(\widehat{\mathcal{A}})} = \|a\|_{\mathcal{A}}$ holds for all $a, b, c, d \in \mathcal{A}$ and numbers α, β.*

4. If two CBA $\mathcal{A}$ and $\mathcal{B}$ are isometrically isomorphic (we write $\mathcal{A} \overset{\text{isom}}{=} \mathcal{B}$) via an isometry j, then the dual isometry $j^* : \mathcal{B}' \to \mathcal{A}'$ provides a homeomorphism of their spectra: $j^*\widehat{\mathcal{B}} = \widehat{\mathcal{A}}$. Also, one has $G\mathcal{A} \overset{\text{isom}}{=} G\mathcal{B}$ via the map $j_\sharp : Ga \mapsto (Ga) \circ j^*$.
5. If X is a compact Hausdorff space, then the Dirac measures exhaust the spectrum of $C(X)$, whereas the map $x_0 \mapsto \delta_{x_0}$ provides a canonical homeomorphism from X onto $\widehat{C(X)}$ (we write $X \overset{\text{hom}}{=} \widehat{C(X)}$). For this reason, the algebra $C(X)$ turns out to be identical to its Gelfand transform $GC(X)$.
   The trick used in inverse problems for reconstruction of manifolds is the following. Assume that we managed to determine (via the data R) a CBA $\mathfrak{A}$, which is known to be isometrically isomorphic to $C(\Omega)$, but neither Ω nor the isometry map is given. Then, finding the spectrum $\widehat{\mathfrak{A}}$, we in fact recover Ω up to a homeomorphism: $\Omega \overset{\text{hom}}{=} \widehat{C(\Omega)} \overset{\text{hom}}{=} \widehat{\mathfrak{A}}$, whereas $C(\Omega) \overset{\text{isom}}{=} GC(\Omega) \overset{\text{isom}}{=} G\mathfrak{A}$ does hold. Thus, $\mathfrak{A}$ provides a homeomorphic copy $\widehat{\mathfrak{A}}$ of the manifold Ω, as well as a concrete isometric copy $C(\widehat{\mathfrak{A}})$ of the algebra $C(\Omega)$.
6. Let $\mathcal{I}$ be a norm-closed ideal in a BA $\mathcal{A}$, $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ the projection "element $\mapsto$ equivalence class". The factor-space $\mathcal{A}/\mathcal{I}$ is endowed with a BA-structure by $\alpha\pi a + \beta\pi b + \pi c\, \pi d := \pi(\alpha a + \beta b + cd)$ for elements $a, b, c, d \in \mathcal{A}$ and numbers α, β; $\|\pi a\| := \inf_{a'\in\pi(a)} \|a'\|$ [9,10].
7. For a BA $\mathcal{A}$ and a subset $S \subset \mathcal{A}$, by $\vee S$ we denote the minimal norm-closed subalgebra of $\mathcal{A}$ that contains S. We say that S *generates* $\vee S$.

Note that the algebras, which will be used for reconstruction, are the *operator C^*-algebras*, i.e., subalgebras of the bounded operator algebra on a Hilbert space closed with respect to the operator conjugation [9].

3.2. The algebra $\mathfrak{T}$

Now let X be the Riemannian manifold Ω under consideration, which is definitely a compact Hausdorff space. Let $C(\Omega)$ be the CBA of real continuous functions on Ω.

The following fact is a straightforward consequence of the separating property of eikonals (Lemma 1) and the Stone–Weierstrass Theorem for *real* algebras [10].

Proposition 3. *The eikonals τ_σ generate $C(\Omega)$: the equality $\vee\{\tau_\sigma \mid \sigma \in \mathcal{R}(\Gamma)\} = C(\Omega)$ holds.*

Recall that $\mathcal{H} = L_2(\Omega)$, $\mathfrak{B}(\mathcal{H})$ is the bounded operator algebra, $\check{\tau}_\sigma \in \mathfrak{B}(\mathcal{H})$ is the multiplication by τ_σ (see Section 2.2). Introduce the (sub)algebra

$$\mathfrak{T} := \vee\{\check{\tau}_\sigma \mid \sigma \in \mathcal{R}(\Gamma)\} \subset \mathfrak{B}(\mathcal{H}) \tag{3.1}$$

generated by scalar operator eikonals. As easily follows from (2.6) and Proposition 3, the map $C(\Omega) \ni \tau_\sigma \mapsto \check{\tau}_\sigma \in \mathfrak{T}$, which connects the generators, is extended to an isometric isomorphism of the CBA $C(\Omega)$ and $\mathfrak{T}$. With regard to items 4, 5 of Section 3.1, the isometry implies

$$\Omega \overset{\text{hom}}{=} \widehat{C(\Omega)} \overset{\text{hom}}{=} \widehat{\mathfrak{T}}. \tag{3.2}$$

On the reconstruction

Here we prepare a fragment of the procedure, which will be used for solving inverse problems.

Assume that we are given a Hilbert space $\tilde{\mathcal{H}} = U\mathcal{H}$, where U is a unitary operator. Also assume that we know the map

$$\mathcal{R}(\Gamma) \times [0, T] \ni \{\sigma, s\} \mapsto \tilde{X}^s_\sigma \in \mathfrak{B}(\tilde{\mathcal{H}}) \quad (T > \operatorname{diam} \Omega), \tag{3.3}$$

where $\tilde{X}^s_\sigma := UX^s_\sigma U^*$, but the operator $U : \mathcal{H} \to \tilde{\mathcal{H}}$ *is unknown.*[5] Show that this map determines the manifold Ω up to isometry. Indeed,

1. using the map, one can construct the operators
$$\tau'_\sigma := \int_0^T s\, d\tilde{X}^s_\sigma = \int_0^T s\, d\,[UX^s_\sigma U^*] \overset{(2.7)}{=} U\check{\tau}_\sigma U^*;$$
2. one can determine the algebra $\tilde{\mathfrak{T}} = \vee\{\tau'_\sigma \mid \sigma \in \mathcal{R}(\Gamma)\} \subset \mathfrak{B}(\tilde{\mathcal{H}})$, which is isometric to $\mathfrak{T} \subset \mathfrak{B}(\mathcal{H})$ (via the *unknown* U);
3. applying the Gelfand transform to $\tilde{\mathfrak{T}}$, one can find its spectrum $\widehat{\tilde{\mathfrak{T}}} =: \tilde{\Omega}$ and the functions $\tilde{\tau}_\sigma := G\tau'_\sigma$ on $\tilde{\Omega}$.

Since $\tilde{\mathfrak{T}} \overset{\text{isom}}{=} \mathfrak{T}$, one has $\tilde{\Omega} := \widehat{\tilde{\mathfrak{T}}} \overset{\text{hom}}{=} \widehat{\mathfrak{T}} \overset{\text{hom}}{=} \Omega$ (see (3.2)). Hence, we get a homeomorphic copy $\tilde{\Omega}$ of the original Ω along with the images $\tilde{\tau}_\sigma$ of the original eikonals τ_σ on Ω.[6] Thus, we have a version of the map (2.3), which determines the copy $\tilde{\Omega}$ (see Proposition 1).

Summarizing, we arrive at the following assertion.

Proposition 4. *The map (3.3) determines the copy $\tilde{\Omega}$ and, hence, determines Ω up to isometry of Riemannian manifolds.*

Moreover, the procedure 1–3 provides the copy $\tilde{\Omega}$.

3.3. The algebra $\mathfrak{E}$

Recall that the eikonals ε_σ are introduced on a 3d-manifold Ω by (2.13).

An operator (sub)algebra

$$\mathfrak{E} := \vee\{\varepsilon_\sigma \mid \sigma \in \mathcal{R}(\Gamma)\} \subset \mathfrak{B}(\mathcal{C}) \tag{3.4}$$

is a "solenoidal" analog of the algebra $\mathfrak{T}$ defined by (3.1). It is a real algebra generated by self-adjoint operators.[7] In contrast to $\mathfrak{T}$, the algebra $\mathfrak{E}$ is not commutative (see the remark at the end of Section 2.3). However, this non-commutativity is weak in the following sense.

Let $\mathfrak{K} \subset \mathfrak{B}(\mathcal{C})$ be the ideal of compact operators. Denote $\mathfrak{K}[\mathfrak{E}] := \mathfrak{K} \cap \mathfrak{E}$ and $\dot{\mathfrak{E}} := \mathfrak{E}/\mathfrak{K}[\mathfrak{E}]$; let $\pi : \mathfrak{B}(\mathcal{C}) \to \mathfrak{B}(\mathcal{C})/\mathfrak{K}$ be the canonical projection. By (3.4), the latter factor-algebra is generated by equivalence classes of eikonals:

$$\dot{\mathfrak{E}} := \vee\{\pi\varepsilon_\sigma \mid \sigma \in \mathcal{R}(\Gamma)\}.$$

Recall that the eikonals τ_σ generate the algebra $C(\Omega)$: see Proposition 3.

Theorem 3. *The algebra $\dot{\mathfrak{E}}$ is commutative. The map*

$$C(\Omega) \ni \tau_\sigma \mapsto \pi\varepsilon_\sigma \in \dot{\mathfrak{E}} \quad (\sigma \in \mathcal{R}(\Gamma)),$$

which relates the generators, can be extended to an isometric isomorphism from $C(\Omega)$ onto $\dot{\mathfrak{E}}$.

---

5 In other words, we are given a representation of the projection family $\{X^s_\sigma\}_{\sigma\in\mathcal{R}(\Gamma)}$ in the space $\tilde{\mathcal{H}}$.

6 By construction, $\tilde{\tau}_\sigma$ turns out to be a pull-back function of τ_σ via the homeomorphism $\tilde{\Omega} \to \Omega$.

7 As such, $\mathfrak{E}$ is a C*-algebra.

The proof is based on Theorem 1 (see Sections 5.2 and 5.3).

With regard to items 4, 5 of Section 3.1, the relation $C(\Omega) \overset{\text{isom}}{=} \dot{\mathfrak{E}}$ established by Theorem 3 implies

$$\Omega \overset{\text{hom}}{=} \widehat{C(\Omega)} \overset{\text{hom}}{=} \widehat{\dot{\mathfrak{E}}}. \tag{3.5}$$

Remark. Examples, in which factorization eliminates noncommutativity, are well known. For instance, let X be a compact smooth manifold (without boundary), and let $\mathfrak{A} \subset \mathfrak{B}(L_2(X))$ be a C*-algebra generated by a certain class of pseudo differential operators of order 0. Then the factor-algebra $\mathfrak{A}/\mathfrak{K}$ is commutative and isomorphic to the algebra of continuous functions on the cosphere bundle of X (see [14]).

On the reconstruction

Here we provide an analog of the procedure described in Section 3.2. This analog is relevant to inverse problems in electrodynamics. Recall that Y^s_σ is the projection in $\mathcal{C}$ onto the subspace $\mathcal{C}\langle\Omega^s[\sigma]\rangle$.

Assume that we are given a Hilbert space $\tilde{\mathcal{C}} = U\mathcal{C}$, where U is a unitary operator. Also assume that we know the map

$$\mathcal{R}(\Gamma) \times [0, T] \ni \{\sigma, s\} \mapsto \tilde{Y}^s_\sigma \in \mathfrak{B}(\tilde{\mathcal{C}}) \quad (T > \operatorname{diam} \Omega), \tag{3.6}$$

where $\tilde{Y}^s_\sigma := UY^s_\sigma U^*$, but the operator $U : \mathcal{C} \to \tilde{\mathcal{C}}$ is *unknown*. Show that this map determines the manifold Ω up to isometry. Indeed,

1. using the map, one can construct the operators
$$\varepsilon'_\sigma := \int_0^T s\, d\tilde{Y}^s_\sigma = \int_0^T s\, d\,[UY^s_\sigma U^*] = U\left[\int_0^T s\, d\, Y^s_\sigma\right] U^* \overset{(2.13)}{=} U\check{\varepsilon}_\sigma U^*;$$
2. determine the algebra $\mathfrak{E}' = \vee\{\varepsilon'_\sigma \mid \sigma \in \mathcal{R}(\Gamma)\} \subset \mathfrak{B}(\tilde{\mathcal{C}})$, which is isometric to $\mathfrak{E} \subset \mathfrak{B}(\mathcal{C})$ (via *unknown* U)
3. construct the factor-algebra $\tilde{\mathfrak{E}} := \mathfrak{E}'/\mathfrak{K}[\mathfrak{E}']$ over the compact operator ideal in $\mathfrak{E}'$. By construction, one has $\tilde{\mathfrak{E}} \overset{\text{isom}}{=} \mathfrak{E}/\mathfrak{K}[\mathfrak{E}] =: \dot{\mathfrak{E}}$.
4. Applying the Gelfand transform to $\tilde{\mathfrak{E}}$, one can find its spectrum $\widehat{\tilde{\mathfrak{E}}} =: \tilde{\Omega}$ and the functions $\tilde{\tau}_\sigma := G\pi\,\varepsilon'_\sigma$ on $\tilde{\Omega}$.

Since $\tilde{\mathfrak{E}} \overset{\text{isom}}{=} \dot{\mathfrak{E}}$, one has

$$\tilde{\Omega} := \widehat{\tilde{\mathfrak{E}}} \overset{\text{hom}}{=} \widehat{\dot{\mathfrak{E}}} \overset{\text{hom}}{=} \Omega$$

(see (3.5)). So, we get a homeomorphic copy $\tilde{\Omega}$ of the original Ω, along with the images $\tilde{\tau}_\sigma$ of the original eikonals τ_σ on Ω. Thus, we have a version of the map (2.3). This map determines the Riemannian structure on $\tilde{\Omega}$, which converts it into an isometric copy of Ω (see Proposition 1).

Summarizing, we arrive at the following result.

Proposition 5. *The map (3.6) determines the copy $\tilde{\Omega}$, and thus it determines Ω up to isometry of Riemannian manifolds.*

Moreover, the procedure 1–4 enables one to construct the copy $\tilde{\Omega}$. This procedure differs from its scalar analog by one additional step that is factorization.

4. Inverse problems

4.1. Acoustic system

With the manifold Ω one associates a dynamical system α^T of the form

$$u_{tt} - \Delta u = 0 \quad \text{in } (\Omega \backslash \Gamma) \times (0, T) \tag{4.1}$$

$$u|_{t=0} = u_t|_{t=0} = 0 \quad \text{in } \Omega \tag{4.2}$$

$$u = f \quad \text{on } \Gamma \times [0, T], \tag{4.3}$$

where Δ is the (scalar) Beltrami–Laplace operator, $t = T > 0$ is a final time, f is a *boundary control*, $u = u^f(x, t)$ is a solution. For controls of the smooth class

$$\mathcal{M}^T := \{f \in C^\infty(\Gamma \times [0, T]) \mid \operatorname{supp} f \subset \Gamma \times (0, T]\}$$

problem (4.1)–(4.3) has a unique classical (smooth) solution u^f. Note that the condition on $\operatorname{supp} f$ means that f vanishes near $t = 0$.

From the physical viewpoint, u^f can be interpreted as an acoustic *wave*, which is initiated by the boundary sound source f and propagates into a domain Ω filled with an inhomogeneous medium.

Attributes

- The space of controls $\mathcal{F}^T := L_2(\Gamma \times [0, T])$ is said to be an *outer space* of the system α^T. The smooth class $\mathcal{M}^T$ is dense in $\mathcal{F}^T$.

The outer space contains the subspaces

$$\mathcal{F}_\sigma^{T,s} := \{f \in \mathcal{F}^T \mid \operatorname{supp} f \subset \sigma \times [T-s, T]\}, \quad \sigma \in \mathcal{R}(\Gamma).$$

Such a subspace consists of controls that are located on σ and are switched on with delay $T-s$ (the quantity s is an action time).

- An *inner space* of the system is $\mathcal{H} = L_2(\Omega)$. The waves $u^f(\cdot, t)$ are time dependent elements of $\mathcal{H}$.
- In the system α^T, the input $\mapsto$ state correspondence is realized by a *control operator* $W^T : \mathcal{F}^T \to \mathcal{H}$, $\operatorname{Dom} W^T = \mathcal{M}^T$,

$$W^T f := u^f(\cdot, T).$$

A specific feature of the system governed by the *scalar* wave equation (4.1) is that W^T is a bounded operator. Therefore one can extend it from $\mathcal{M}^T$ onto $\mathcal{F}^T$ by continuity, which we assume to be done.

- The input $\mapsto$ output map is represented by a *response operator* $R^T : \mathcal{F}^T \to \mathcal{F}^T$, $\operatorname{Dom} R^T = \mathcal{M}^T$,

$$R^T f := u_\nu^f\big|_{\Gamma \times [0,T]},$$

where $(\cdot)_\nu$ is the derivative with respect to the outward normal.

The following obvious fact was already mentioned in the Introduction.

Proposition 6. *If two Riemannian manifolds have the mutual boundary and are isometric (the isometry being the identity on the boundary), then their (acoustic) response operators coincide. In particular, for the manifold Ω and its copy $\tilde{\Omega}$ one has $R^{2T} = \tilde{R}^{2T}$ for any $T > 0$.*

- A *connecting operator* $C^T : \mathcal{F}^T \to \mathcal{F}^T$ is defined by

$$C^T := (W^T)^* W^T. \tag{4.4}$$

The definition implies

$$(C^T f, g)_{\mathcal{F}^T} = (W^T f, W^T g)_{\mathcal{H}} = \left(u^f(\cdot, T), u^g(\cdot, T)\right)_{\mathcal{H}},$$

i.e., C^T connects the Hilbert metrics of the outer and inner spaces. A significant fact is that the connecting operator is determined by the response operator of the system α^{2T} through an explicit formula

$$C^T = 2^{-1}(S^T)^* R^{2T} J^{2T} S^T, \tag{4.5}$$

where the map $S^T : \mathcal{F}^T \to \mathcal{F}^{2T}$ extends the controls from $\Gamma \times [0, T]$ to $\Gamma \times [0, 2T]$ as odd functions (of time t) with respect to $t = T$; $J^{2T} : \mathcal{F}^{2T} \to \mathcal{F}^{2T}$ is an integration: $(J^{2T} f)(\cdot, t) = \int_0^t f(\cdot, s)\, ds$ (see [6,7]).

Controllability

The set $\mathcal{U}_\sigma^s := \{u^f(\cdot, s) \mid f \in \mathcal{F}_\sigma^T\}$ is said to be *reachable* (from σ, at the moment $t = s$).

The operator Δ, which governs the evolution of the system α^T, does not depend on time. For this reason, a time delay of controls implies the same delay of the waves. As a result, one has

$$\mathcal{U}_\sigma^s = W^T \mathcal{F}_\sigma^{T,s}, \quad 0 \leqslant s \leqslant T.$$

Problem (4.1)–(4.3) is hyperbolic and the finiteness of domains of influence does hold for its solutions: for delayed controls one has

$$\operatorname{supp} u^f(\cdot, T) \subset \overline{\Omega^s[\sigma]}, \quad f \in \mathcal{F}_\sigma^{T,s}. \tag{4.6}$$

The latter means that in the system α^T the waves propagate with the unit velocity. As a result, the embedding $\mathcal{U}_\sigma^s \subset \mathcal{H}\langle\Omega^s[\sigma]\rangle$ is valid. The character of this embedding is of principal importance: it turns out to be *dense*. The following result is based upon the fundamental Holmgren–John–Tataru uniqueness theorem (see [6,7] for details).

Proposition 7. *For any $s > 0$ and $\sigma \in \mathcal{R}(\Gamma)$, the relation $\overline{\mathcal{U}_\sigma^s} = \mathcal{H}\langle\Omega^s[\sigma]\rangle$ is valid (the closure in $\mathcal{H}$). In particular, for $s = T > \operatorname{diam} \Omega$ one has $\overline{\mathcal{U}_\sigma^T} = \mathcal{H}$.*

In control theory this property is referred to as *local approximate boundary controllability* of the system α^T. It shows that the reachable sets are rich enough: any function supported in the neighborhood $\Omega^s[\sigma]$ can be approximated (in $\mathcal{H}$-metric) by a wave $u^f(\cdot, T)$ by means of the proper choice of a control $f \in \mathcal{F}_\sigma^{T,s}$.

By P_σ^s we denote the projection in $\mathcal{H}$ onto the reachable subspace $\overline{\mathcal{U}_\sigma^s}$ and call it a *wave projection*. Recall that X_σ^s is the projection in $\mathcal{H}$ onto $\mathcal{H}\langle\Omega^s[\sigma]\rangle$, which cuts off functions onto the neighborhood $\Omega^s[\sigma]$. As a consequence of the Proposition 7, we obtain

$$P_\sigma^s = X_\sigma^s, \quad s > 0, \ \sigma \in \mathcal{R}(\Gamma). \tag{4.7}$$

4.2. IP of acoustics

Setup

The dynamical inverse problem (IP) for system (4.1)–(4.3) is:

for a fixed $T > \operatorname{diam} \Omega$, given the response operator R^{2T}, recover the manifold Ω.

A physical meaning of the condition $T > \operatorname{diam}\Omega$ is that the waves u^f, which prospect the manifold from the parts σ of its boundary, need time large enough to pass through the whole Ω: see (4.6) and (2.1).

As was clarified in the Introduction, *to recover* Ω means to construct (via a given R^{2T}) a Riemannian manifold, which has the same boundary Γ, and possesses the response operator, which is equal to R^{2T}. Anticipating things, we claim that R^{2T} determines the copy $\tilde\Omega$. Thus, $\tilde\Omega$ provides the solution to the IP.

Model

As an operator connecting two Hilbert spaces, the control operator $W^T : \mathcal{F}^T \to \mathcal{H}$ can be represented in the form of a *polar decomposition*

$$W^T = \Phi^T |W^T|,$$

where

$$|W^T| := \left[\left(W^T\right)^* W^T\right]^{\frac12} \overset{(4.4)}{=} \left(C^T\right)^{\frac12}$$

and $\Phi^T : |W^T| f \mapsto W^T f$ is an isometry from $\operatorname{Ran}|W^T| \subset \mathcal{F}^T$ onto $\operatorname{Ran} W^T \subset \mathcal{H}$ (see, e.g., [15]). In what follows we assume that Φ^T is extended by continuity to an isometry from $\overline{\operatorname{Ran}|W^T|}$ onto $\overline{\operatorname{Ran} W^T}$.

Recall that $\mathcal{U}^s_\sigma := W^T \mathcal{F}^{T,s}_\sigma$ are reachable sets of the system α^T and P^s_σ is the projection in $\mathcal{H}$ onto $\overline{\mathcal{U}^s_\sigma}$.

We say that the (sub)space $\tilde{\mathcal{H}} := \overline{\operatorname{Ran}|W^T|} \subset \mathcal{F}^T$ is a *model inner space*, and $\tilde{\mathcal{U}}^s_\sigma := |W^T| \mathcal{F}^{T,s}_\sigma \subset \tilde{\mathcal{H}}$ is a *model reachable set*. By $\tilde P^s_\sigma$ we denote the projection in $\tilde{\mathcal{H}}$ onto $\overline{\tilde{\mathcal{U}}^s_\sigma}$ and call it a *model wave projection*.

The model and original objects are related via the isometry Φ^T. In particular, the definitions imply $\Phi^T \tilde P^s_\sigma = P^s_\sigma \Phi^T$.

Now let $T > \operatorname{diam}\Omega$, so that $\Omega^T[\sigma] = \Omega$ holds for any σ. By Proposition 7, one has $\overline{\operatorname{Ran} W^T} = \mathcal{H}$. For this reason, the isometry Φ^T turns out to be a unitary operator from $\tilde{\mathcal{H}}$ onto $\mathcal{H}$. Its inverse $U := (\Phi^T)^*$ maps $\mathcal{H}$ onto $\tilde{\mathcal{H}}$ isometrically and $UP^s_\sigma = \tilde P^s_\sigma U$ holds.

Let $\tilde X^s_\sigma := U X^s_\sigma U^*$ be the image (in $\tilde{\mathcal{H}}$) of the cutting-off projection. The property (4.7) implies that

$$\tilde P^s_\sigma = \tilde X^s_\sigma, \quad s > 0,\ \sigma \in \mathcal{R}(\Gamma). \tag{4.8}$$

Solving IP

It suffices to show that the operator R^{2T} determines the copy $\tilde\Omega$. One can do this by the following procedure.

1. Find the connecting operator by (4.5). Determine the operator $|W^T| = \left(C^T\right)^{\frac12}$ and the subspace $\tilde{\mathcal{H}} = \overline{\operatorname{Ran}|W^T|} \subset \mathcal{F}^T$.
2. Fix a $\sigma \in \mathcal{R}(\Gamma)$ and $s \in (0, T]$. In $\tilde{\mathcal{H}}$ recover the model reachable set $\tilde{\mathcal{U}}^s_\sigma = |W^T| \mathcal{F}^{T,s}_\sigma \subset \tilde{\mathcal{H}}$ and determine the corresponding projection $\tilde P^s_\sigma$. By (4.8), we get the projection $\tilde X^s_\sigma$. Thus, the map (3.3) is at our disposal.
3. By Proposition 4, this map determines the copy $\tilde\Omega$. Its response operator $\tilde R^{2T}$ coincides with the given R^{2T}: see Proposition 6.

The acoustic IP is solved.

4.3. Maxwell system

Here Ω is a smooth, compact, connected, and oriented Riemannian 3d-manifold with the boundary Γ.

Propagation of electromagnetic waves in a curved space is described by the dynamical Maxwell system α^T_{M}

$$e_t = \operatorname{curl} h, \qquad h_t = -\operatorname{curl} e \quad \text{in } (\Omega\backslash\Gamma) \times (0, T), \tag{4.9}$$

$$e|_{t=0} = 0, \qquad h|_{t=0} = 0 \quad \text{in } \Omega, \tag{4.10}$$

$$e_\theta = f \quad 0 \leqslant t \leqslant T, \tag{4.11}$$

where $e_\theta := e - e \cdot \nu\, \nu$ is a tangent component of e on the boundary, f is a time-dependent element of $T\Gamma$ (*boundary control*), e and h are the electric and magnetic components of the solution. Let

$$\mathcal{M}^T := \left\{ f \in C^\infty\left([0, T]; \vec C^\infty(\Gamma)\right) \,\middle|\, \operatorname{supp} f \subset (0, T] \right\}$$

be the class of smooth controls vanishing near $t = 0$. For $f \in \mathcal{M}^T$, problem (4.9)–(4.11) has a unique classical smooth solution $\{e^f(x, t), h^f(x, t)\}$.

Since a divergence is an integral of motion of the Maxwell system, (4.10) implies

$$\operatorname{div} e^f(\cdot, t) = 0, \qquad \operatorname{div} h^f(\cdot, t) = 0, \quad t \geqslant 0.$$

Attributes

- The space of controls $\mathcal{F}^T := L_2\left([0, T]; \vec L_2(\Gamma)\right)$ is the *outer space* of the system α^T_{M}. The smooth class $\mathcal{M}^T$ is dense in $\mathcal{F}^T$.

  Denote $\vec L_2(\sigma) := \{a \in \vec L_2(\Gamma) \mid \operatorname{supp} a \subseteq \sigma\}$. The outer space contains the subspaces

  $$\mathcal{F}^{T,s}_\sigma := \left\{ f \in L_2\left([0, T]; \vec L_2(\sigma)\right) \mid \operatorname{supp} f \subseteq [T - s, T] \right\} \quad \sigma \in \mathcal{R}(\Gamma)$$

  of controls, which are located on σ and switched on with delay $T - s$ (s is the action time).

- The *inner space* of the system is the space $\mathcal{C} \oplus \mathcal{C}$. By (4.9), the solutions $\{e^f(\,\cdot\,,t), h^f(\,\cdot\,,t)\}$ are time dependent elements of this space. Also, we select its electric part $\mathcal{C} \oplus \{0\} \ni e^f(\,\cdot\,,t)$.
- The input $\mapsto$ state correspondence is realized by a *control operator* $W^T_{\rm M} : \mathcal{F}^T \to \mathcal{C} \oplus \mathcal{C}$, $\operatorname{Dom} W^T_{\rm M} = \mathcal{M}^T$, $W^T_{\rm M} f := \{e^f(\,\cdot\,,T), h^f(\,\cdot\,,T)\}$. Its electric part is $W^T : \mathcal{F}^T \to \mathcal{C}$,

$$W^T : f \mapsto e^f(\,\cdot\,,T).$$

In contrast to the acoustic (scalar) system, $W^T_{\rm M}$ and W^T are unbounded (but closable) operators.

A reason to select the electric part of the system $\alpha^T_{\rm M}$ is that it is the electric component, which is controlled on the boundary: see (4.11). For this reason, e^f and h^f are not quite independent. Moreover, for $T < \inf\{r > 0 \mid \Omega^r[\Gamma] = \Omega\}$ the operator W^T is injective and, hence, $e^f(\,\cdot\,,T)$ determines $h^f(\,\cdot\,,T)$ [7,11].
- The input $\mapsto$ output map of the system $\alpha^T_{\rm M}$ is represented by a *response operator* $R^T : \mathcal{F}^T \to \mathcal{F}^T$, $\operatorname{Dom} R^T = \mathcal{M}^T$,

$$R^T f := \nu \wedge h^f\big|_{\Gamma \times [0,T]}.$$

The following fact is quite obvious.

Proposition 8. *Let two Riemannian manifolds have the mutual boundary and be isometric, and let the isometry be the identity on the boundary. Then their Maxwell response operators coincide. In particular, for the manifold Ω and its canonical copy $\tilde\Omega$ one has $R^{2T} = \tilde R^{2T}$ for any $T > 0$.*

- An electric *connecting operator* $C^T : \mathcal{F}^T \to \mathcal{F}^T$ is introduced via a *connecting form* c^T, $\operatorname{Dom} c^T = \mathcal{M}^T \times \mathcal{M}^T$,

$$c^T[f,g] := \left(e^f(\,\cdot\,,T), e^g(\,\cdot\,,T)\right)_{\mathcal{C}} = \left(W^T f, W^T g\right)_{\mathcal{C}}.$$

It is a Hermitian nonnegative bilinear form. As such, it is closable, the closure $\bar c^T$ being defined on $\mathcal{N}^T \times \mathcal{N}^T$, where $\mathcal{N}^T$ is a lineal set in $\mathcal{F}^T$, $\mathcal{N}^T \supset \mathcal{M}^T$. The form $\bar c^T$ determines a unique self-adjoint operator C^T by the relation

$$(C^T f, g)_{\mathcal{F}^T} = \bar c^T[f,g], \quad f \in \operatorname{Dom} C^T,\ g \in \mathcal{N}^T$$

(see, e.g., [15]). In fact, to close c^T is to close W^T, and one has $\mathcal{N}^T = \operatorname{Dom} \bar W^T = \operatorname{Dom} (C^T)^{\frac12}$. Hence, the knowledge of c^T enables one to extend W^T from $\mathcal{M}^T$ to $\mathcal{N}^T$. In what follows this extension (closure) is assumed to be performed and is denoted by the same symbol W^T. The images $W^T f$ for $f \in \mathcal{N}^T$ are regarded as generalized solutions $e^f(\,\cdot\,,T)$.

As a result, one has the relations

$$\bar c^T[f,g] = \left((C^T)^{\frac12} f, (C^T)^{\frac12} g\right)_{\mathcal{F}^T} = \left(W^T f, W^T g\right)_{\mathcal{C}}, \quad f,g \in \mathcal{N}^T. \tag{4.12}$$

The key fact is that the connecting form is determined by the response operator of the system $\alpha^{2T}_{\rm M}$ via an explicit formula

$$c^T[f,g] = 2^{-1}\left((S^T)^* R^{2T} J^{2T} S^T f, g\right)_{\mathcal{F}^T}, \quad f,g \in \mathcal{M}^T, \tag{4.13}$$

where the map $S^T : \mathcal{F}^T \to \mathcal{F}^{2T}$ extends the controls from $\Gamma \times [0,T]$ to $\Gamma \times [0,2T]$ as odd functions (of time t) with respect to $t = T$; $J^{2T} : \mathcal{F}^{2T} \to \mathcal{F}^{2T}$ is an integration: $(J^{2T} f)(\cdot,t) = \int_0^t f(\cdot,s)\,ds$ (see [7]).

Summing up aforesaid, we can claim that R^{2T} determines the operator $(C^T)^{\frac12}$ in accordance with the scheme

$$R^{2T} \overset{(4.13)}{\Rightarrow} c^T \Rightarrow \bar c^T \Rightarrow C^T \Rightarrow (C^T)^{\frac12}. \tag{4.14}$$

Controllability

The set $\mathcal{E}^s_\sigma := \{e^f(\,\cdot\,,s) \mid f \in \mathcal{F}^T_\sigma \cap \mathcal{M}^T\}$ is said to be *reachable* (from σ, at the moment $t = s$).

The operators curl , which govern the evolution of the system $\alpha^T_{\rm M}$, do not depend on time. For this reason, a time delay of controls implies the same delay of the waves. As a result, one can represent

$$\mathcal{E}^s_\sigma = W^T\left[\mathcal{F}^{T,s}_\sigma \cap \mathcal{M}^T\right].$$

The Maxwell system (4.9)–(4.11) obeys the finiteness of domains of influence principle: for the delayed controls one has

$$\operatorname{supp} e^f(\,\cdot\,,T) \subset \overline{\Omega^s[\sigma]}, \qquad f \in \left[\mathcal{F}^{T,s}_\sigma \cap \mathcal{M}^T\right]. \tag{4.15}$$

The latter means that electromagnetic waves propagate with the unit velocity. As a consequence, the embedding $\mathcal{E}^s_\sigma \subset \mathcal{C}\langle\Omega^s[\sigma]\rangle$ is valid. Moreover, this embedding is *dense*. This fact is derived from a vectorial version of the Holmgren–John–Tataru uniqueness theorem (see [7] for detail).

Proposition 9. *For any $s > 0$ and $\sigma \in \mathcal{R}(\Gamma)$, the relation $\overline{\mathcal{E}^s_\sigma} = \mathcal{C}\langle\Omega^s[\sigma]\rangle$ is valid (the closure in $\mathcal{C}$). In particular, for $s = T > \operatorname{diam}\Omega$ one has $\overline{\mathcal{E}^T_\sigma} = \mathcal{C}$.*

This property is interpreted as *local approximate boundary controllability* of the electric subsystem of $\alpha^T_{\rm M}$.

By E^s_σ we denote the projection in $\mathcal{C}$ onto the reachable subspace $\overline{\mathcal{E}^s_\sigma}$ and call it a *wave projection*. Recall that Y^s_σ is the projection in $\mathcal{C}$ onto $\mathcal{C}\langle\Omega^s[\sigma]\rangle$. As a consequence of the Proposition 9, we obtain

$$E^s_\sigma = Y^s_\sigma, \quad s > 0,\ \sigma \in \mathcal{R}(\Gamma). \tag{4.16}$$

4.4. IP of electrodynamics

Setup

The dynamical inverse problem (IP) for system (4.9)–(4.11) is:

for a fixed $T > \operatorname{diam} \Omega$, *given the response operator* R^{2T}, *recover the manifold* Ω.

The physical meaning of the condition $T > \operatorname{diam} \Omega$ is the same as in the acoustical case: the electromagnetic waves need sufficiently large time to prospect the whole Ω: see (4.15) and (2.1).

As before, *to recover* Ω means to construct (via given R^{2T}) a Riemannian manifold, which has the same boundary Γ, and possesses the response operator, which is equal to R^{2T}. As well as in the scalar case, we will show that R^{2T} determines the copy $\tilde{\Omega}$. Thus, $\tilde{\Omega}$ will provide the solution to the IP.

Model

Representing the (closed) control operator $W^T : \mathcal{F}^T \to \mathcal{C}$ in the polar decomposition form, one has $W^T = \Psi^T|W^T|$, where $|W^T| := \left[(W^T)^* W^T\right]^{\frac{1}{2}}$ and $\Psi^T : |W^T|f \mapsto W^T f$ is an isometry from $\operatorname{Ran}|W^T| \subset \mathcal{F}^T$ onto $\operatorname{Ran} W^T \subset \mathcal{C}$ [15]. In what follows Ψ^T is assumed to be extended by continuity to an isometry from $\overline{\operatorname{Ran}|W^T|}$ onto $\overline{\operatorname{Ran} W^T}$. Also note that (4.12) implies $|W^T| = (C^T)^{\frac{1}{2}}$.

Recall that $\mathcal{E}^s_\sigma := W^T[\mathcal{F}^{T,s}_\sigma \cap \mathcal{M}^T]$ is an electric reachable set and E^s_σ is the (wave) projection in $\mathcal{C}$ onto $\overline{\mathcal{E}^s_\sigma}$.

Let us say that (sub)space $\tilde{\mathcal{C}} := \overline{\operatorname{Ran}|W^T|} \subset \mathcal{F}^T$ is a *model inner space*, $\tilde{\mathcal{E}}^s_\sigma := |W^T|\left[\mathcal{F}^{T,s}_\sigma \cap \mathcal{M}^T\right] \subset \tilde{\mathcal{C}}$ are *model reachable sets*. By $\tilde{E}^s_\sigma$ we denote the projection in $\tilde{\mathcal{C}}$ onto $\overline{\tilde{\mathcal{E}}^s_\sigma}$ and call it a *model wave projection*.

The model and original objects are related via the isometry Ψ^T. In particular, the definitions imply that $\Psi^T\tilde{E}^s_\sigma = E^s_\sigma\Psi^T$.

Now, let $T > \operatorname{diam} \Omega$. By Proposition 9, one has $\overline{\operatorname{Ran} W^T} = \mathcal{C}$. Therefore the isometry Ψ^T turns out to be a unitary operator from $\tilde{\mathcal{C}}$ onto $\mathcal{C}$. Its inverse $U := (\Psi^T)^*$ maps $\mathcal{C}$ onto $\tilde{\mathcal{C}}$ isometrically, and $UE^s_\sigma = \tilde{E}^s_\sigma U$ holds.

Let $\tilde{Y}^s_\sigma := UY^s_\sigma U^*$. The property (4.16) implies that

$$\tilde{E}^s_\sigma = \tilde{Y}^s_\sigma, \quad s > 0, \ \sigma \in \mathcal{R}(\Gamma). \tag{4.17}$$

Solving IP

Let us show that the operator R^{2T} determines the copy $\tilde{\Omega}$.

1. Find the connecting form c^T by (4.13). Determine the model control operator $|W^T| = \left(C^T\right)^{\frac{1}{2}}$ (see (4.14)) and the model inner space $\tilde{\mathcal{C}} = \overline{\operatorname{Ran}|W^T|} \subset \mathcal{F}^T$.
2. Fix a $\sigma \in \mathcal{R}(\Gamma)$ and $s \in (0, T)$. In $\tilde{\mathcal{C}}$ recover the model reachable set $\tilde{\mathcal{E}}^s_\sigma = |W^T|\left[\mathcal{F}^{T,s}_\sigma \cap \mathcal{M}^T\right] \subset \tilde{\mathcal{C}}$ and determine the corresponding projection $\tilde{E}^s_\sigma$. By (4.17), we get the projection $\tilde{Y}^s_\sigma$. Thus, the map (3.6) is at our disposal.
3. By Proposition 5, this map determines the copy $\tilde{\Omega}$. Its Maxwell response operator $\tilde{R}^{2T}$ coincides with the given R^{2T} (see Proposition 8).

The IP of electrodynamics is solved.

4.5. Comments

- In this paper, the condition $T > \operatorname{diam} \Omega$ is imposed for the sake of simplicity. It provides the embedding $\check{\tau}_\sigma C(\Omega) \subset C(\Omega)$, which is convenient just for technical reasons. However, there is a *time-optimal* setup of the reconstruction problem, which takes into account a local character of dependence of the acoustic and Maxwell response operators on a near-boundary part of the manifold. Namely, by the finiteness of the influence domain, for an arbitrary fixed $T > 0$ the operator R^{2T} is determined by the submanifold $\Omega^T[\Gamma]$ (does not depend on the part $\Omega \backslash \Omega^T[\Gamma]$). Therefore, the natural setup is: *for a fixed* $T > 0$, *given the operator* R^{2T}, *recover* $\Omega^T[\Gamma]$. In such a stronger form the problem is solved in [7,16].
- In reconstruction via a spectral triple $\{\mathcal{A}, \mathcal{H}, \mathcal{D}\}$ (see [1,3]), the algebra provides a topological space (which is $\widehat{\mathcal{A}}$), whereas the operator $\mathcal{D}$ encodes a Riemannian metric on $\widehat{\mathcal{A}}$. The metric is recovered (via $\mathcal{D}$) by means of the *Connes distance formula*. In our scheme, the object responsible for the metric is a selected family of generators of the algebra (which is the set of eikonals).
- Dealing with the reconstruction problem for a *metric graph* Ω, one can introduce a straightforward analog of the eikonal algebra $\mathfrak{T}$ [17]. However, this algebra turns out to be strongly noncommutative: no factorization converts $\mathfrak{T}$ into a commutative algebra. For this reason, we should deal with its *Jacobson spectrum* $\widehat{\mathfrak{T}}$, which is the topologized set of the primitive ideals of $\mathfrak{T}$ [9]. As known examples show, the structure of $\mathfrak{T}$ is related with the geometry of Ω but in a rather implicit way. An intriguing fact is that in some examples, the space $\widehat{\mathfrak{T}}$ is non-Hausdorff. It may contain "clusters", which are the groups of nonseparable points. Presumably, the clusters of $\widehat{\mathfrak{T}}$ are related to interior vertices of the graph. The reconstruction $R \Rightarrow \Omega$ for graphs is yet an open challenging problem, and we hope for our "algebraic approach".

5. Proofs of theorems

In what follows $(\cdot,\cdot)_U$ and $\|\cdot\|_U$ denote the inner product and the norm in $L_2(U)$ or $\vec{L}_2(U)$. In this section we consider X^s_σ as the projection in $\vec{\mathcal{H}}$, which cuts off fields to $\Omega^s[\sigma]$. Similarly we consider $\check{\tau}_\sigma$ as operator acting in $\vec{\mathcal{H}}$ by the rule (2.5).

5.1. Proof of Theorem 1

Let a field $z \in \vec{\mathcal{H}}$ satisfy the condition $\operatorname{curl} z \in \vec{\mathcal{H}}$. Following [18], we say that the field z satisfies the condition

$$\nu \wedge z|_\Gamma = 0 \tag{5.1}$$

if for any field $v \in \vec{\mathcal{H}}$, such that $\operatorname{curl} v \in \vec{\mathcal{H}}$, we have

$$(z, \operatorname{curl} v)_\Omega = (\operatorname{curl} z, v)_\Omega. \tag{5.2}$$

Remark. It can be shown that by the smoothness of the boundary Γ it suffices to check this condition for $v \in \vec{C}^\infty(\Omega)$ only.

Introduce the space

$$F := \{u \in \vec{\mathcal{H}} : \operatorname{div} u \in L_2(\Omega),\ \operatorname{curl} u \in \vec{\mathcal{H}},\ \nu \wedge u|_\Gamma = 0\}$$

with the norm

$$\|u\|^2_F := \|u\|^2_\Omega + \|\operatorname{div} u\|^2_\Omega + \|\operatorname{curl} u\|^2_\Omega.$$

The following result is valid for an $\Omega \subset \mathbb{R}^3$ (see [18, Section 8.4]) and can be easily generalized to a smooth manifold.

Theorem 4. *The embedding of the space F in $\vec{\mathcal{H}}$ is compact.*

Actually, the stronger fact holds true: the space F coincides with the vector Sobolev space $\vec{H}^1(\Omega)$, which is compactly embedded in $\vec{\mathcal{H}}$. However, Theorem 4 will suffice for our purposes. Theorem 4 is used in spectral analysis of the Maxwell operator on compact manifolds (see, e.g., [19]).

Let us outline the proof of Theorem 1. We obtain estimates for L_2-norms of the curl and divergence of the difference $\check{\tau}_\sigma u - \varepsilon_\sigma u$ by the L_2-norm of $u \in \mathcal{C}$ (inequalities (5.16), (5.20)), and establish the boundary condition (5.1) on Γ for this difference. This means that the field $\check{\tau}_\sigma u - \varepsilon_\sigma u$ belongs to F with the corresponding norm estimate, which implies that the operator $\check{\tau}_\sigma - \varepsilon_\sigma$ restricted to $\mathcal{C}$ is compact (by compactness of the embedding $F \subset \vec{\mathcal{H}}$).

We will use the following relations, which are valid for any $T > 0$:

$$\int_{[0,T]} s\, dX^s_\sigma = T X^T_\sigma - \int_0^T X^s_\sigma\, ds, \tag{5.3}$$

$$\int_{[0,T]} s\, dY^s_\sigma = T Y^T_\sigma - \int_0^T Y^s_\sigma\, ds. \tag{5.4}$$

The operator integrals on the right-hand sides are understood as bounded (symmetric) operators (in $\vec{\mathcal{H}}$ for X^s_σ and in $\mathcal{C}$ for Y^s_σ) defined by their bilinear forms

$$\int_0^T ds\, (X^s_\sigma w, z)_\Omega, \quad w, z \in \vec{\mathcal{H}}, \tag{5.5}$$

$$\int_0^T ds\, (Y^s_\sigma u, y)_\Omega, \quad u, y \in \mathcal{C}. \tag{5.6}$$

Both integrals exist since the integrands can be expressed via functions monotonic with respect to s:

$$(X^s_\sigma w, z) = \left[(X^s_\sigma (w+z), w+z) - (X^s_\sigma (w-z), w-z)\right]/4$$

and similarly for Y^s_σ. Note that in bilinear forms (5.5), (5.6) (and thus in the corresponding operator integrals) we do not specify whether the interval of integration contain the end points 0 and T since the integral does not depend on it.

Along with (2.7) relations (5.3), (5.4) imply that for $y \in \mathcal{C}$ we have

$$(\varepsilon_\sigma - \check{\tau}_\sigma)\, y = T(Y^T_\sigma - X^T_\sigma)\, y + \left(\int_0^T X^s_\sigma\, ds - \int_0^T Y^s_\sigma\, ds\right) y.$$

For $T > \operatorname{diam} \Omega$ the first term vanishes, since X^T_σ and Y^T_σ become the identity operators in $\vec{\mathcal{H}}$ and $\mathcal{C}$ correspondingly. Introduce a bounded operator

$$K_\sigma := \int_0^T X^\xi_\sigma\, d\xi - \int_0^T Y^\xi_\sigma\, d\xi$$

acting from $\mathcal{C}$ to $\vec{\mathcal{H}}$. We have

$$(\varepsilon_\sigma - \check{\tau}_\sigma)\, y = K_\sigma y, \quad y \in \mathcal{C}. \tag{5.7}$$

To prove Theorem 1, we need to establish that K_σ is compact.

Although the operator K_σ acts from $\mathcal{C}$ to $\vec{\mathcal{H}}$ and thus cannot be considered as symmetric, it has the following symmetry property:

$$(K_\sigma u, v)_\Omega = (u, K_\sigma v)_\Omega \quad \forall u, v \in \mathcal{C}, \tag{5.8}$$

since

$$\begin{aligned}(K_\sigma u, v)_\Omega &= \int_0^T d\xi\, ((X_\sigma^\xi - Y_\sigma^\xi)\, u, v)_\Omega = \int_0^T d\xi\, (u, (X_\sigma^\xi - Y_\sigma^\xi)\, v)_\Omega \\ &= (u, K_\sigma v)_\Omega .\end{aligned}$$

Define also a family of operators acting from $\mathcal{C}$ to $\vec{\mathcal{H}}$ by

$$K_\sigma^s := \int_0^s X_\sigma^\xi\, d\xi - \int_0^s Y_\sigma^\xi\, d\xi, \quad 0 \leqslant s < \infty.$$

Now we derive the following relation

$$\left(\int_0^s X_\sigma^\xi\, d\xi\ w\right)(x) = \max\{s - \tau_\sigma(x), 0\}\, w(x), \quad x \in \Omega. \tag{5.9}$$

By (5.5) for $w, z \in \vec{\mathcal{H}}$ we have

$$\left(\int_0^s X_\sigma^\xi\, d\xi\ \ w, z\right)_\Omega = \int_0^s (X_\sigma^\xi\, w, z)_\Omega\, d\xi = \int_0^s d\xi \int_\Omega dx\, \chi_{\Omega_\sigma^\xi}\, w \cdot z.$$

Now apply Fubini's theorem

$$\int_0^s d\xi \int_\Omega dx\, \chi_{\Omega_\sigma^\xi}\, w \cdot z = \int_\Omega dx\, w \cdot z \int_0^s d\xi\ \chi_{\Omega_\sigma^\xi}(x).$$

Since the inner integral equals $\max\{s - \tau_\sigma(x), 0\}$, we arrive at the relation

$$\int_\Omega dx\ \max\{s - \tau_\sigma, 0\}\, w \cdot z = (\max\{s - \tau_\sigma, 0\}\, w, z)_\Omega\ .$$

The obtained expression for the bilinear form of the operator integral implies (5.9).

Lemma 2. *Choose $\sigma \subset \Gamma$ and $s > 0$. Let a field $\beta \in \vec{\mathcal{H}}\langle\Omega^s[\sigma]\rangle$ be smooth in $\Omega^s[\sigma]$ (in particular, smooth on the boundary $\Omega^s[\sigma] \cap \Gamma$) and orthogonal to $\mathcal{C}\langle\Omega^s[\sigma]\rangle$. Then for any $z \in \vec{C}^\infty(\Omega)$ one has*

$$(\beta, K_\sigma^s \operatorname{curl} z)_{\Omega^s[\sigma]} = (\beta, \nabla\tau_\sigma \wedge z)_{\Omega^s[\sigma]}.$$

Proof. Let $0 < s' < s$. By the absolute continuity of the Lebesgue integral, we have

$$(\beta, K_\sigma^{s'} \operatorname{curl} z)_{\Omega^{s'}[\sigma]} \to (\beta, K_\sigma^s \operatorname{curl} z)_{\Omega^s[\sigma]}, \quad s' \to s - 0. \tag{5.10}$$

As is obvious, β is orthogonal to $\mathcal{C}\langle\Omega^\xi[\sigma]\rangle$ for $\xi \leqslant s$; therefore,

$$\begin{aligned}(\beta, K_\sigma^{s'} \operatorname{curl} z)_{\Omega^{s'}[\sigma]} &= \int_0^{s'} d\xi\ (\beta, (X_\sigma^\xi - Y_\sigma^\xi) \operatorname{curl} z)_{\Omega^\xi[\sigma]} \\ &= \int_0^{s'} d\xi\ (\beta, X_\sigma^\xi \operatorname{curl} z)_{\Omega^\xi[\sigma]} \overset{(5.9)}{=} (\beta, (s' - \tau_\sigma) \operatorname{curl} z)_{\Omega^{s'}[\sigma]} \\ &= ((s' - \tau_\sigma)\, \beta, \operatorname{curl} z)_{\Omega^{s'}[\sigma]}.\end{aligned}$$

Define the Lipschitz function $h^{s'}$ in Ω as follows:

$$h^{s'}(x) := \max\{s' - \tau_\sigma(x), 0\}.$$

We have

$$((s' - \tau_\sigma)\, \beta, \operatorname{curl} z)_{\Omega^{s'}[\sigma]} = (h^{s'} \beta, \operatorname{curl} z)_\Omega \tag{5.11}$$

(the field $h^{s'}\beta$ is defined in Ω since $h^{s'}$ vanishes outside $\Omega^{s'}[\sigma] \subset \Omega^s[\sigma]$). The field $h^{s'}\beta$ is Lipschitz, as the function $h^{s'}$ is Lipschitz, and the field β is smooth in a neighborhood of $\operatorname{supp} h^{s'}$, so we can apply the formula of integration by parts to the right-hand side in (5.11). The orthogonality of β to $\mathcal{C}\langle\Omega^s[\sigma]\rangle$ implies

$$\operatorname{curl} \beta\, |_{\Omega^s[\sigma]} = 0, \quad \nu \wedge \beta|_{\Omega^s[\sigma] \cap \Gamma} = 0. \tag{5.12}$$

By the second relation we have $\nu \wedge (h^{s'}\beta)|_\Gamma = 0$. So the integral over Γ in the integration by parts vanishes. Applying the first relation in (5.12) and formula (2.10), we obtain

$$(h^{s'}\beta, \operatorname{curl} z)_\Omega = (\operatorname{curl}(h^{s'}\beta), z)_\Omega = (\nabla h^{s'} \wedge \beta, z)_\Omega = ((-\nabla\tau_\sigma) \wedge \beta, z)_{\Omega^{s'}[\sigma]}$$
$$= (\beta, \nabla\tau_\sigma \wedge z)_{\Omega^{s'}[\sigma]}.$$

The latter term tends to $(\beta, \nabla\tau_\sigma \wedge z)_{\Omega^{s}[\sigma]}$ as $s' \to s$. Taking into account (5.10), we obtain the required relation. □

Note that Lemma 2 holds true if $\Omega^s[\sigma] = \Omega$.

Lemma 3. *Let $\sigma \subset \Gamma$. For a field $z \in \vec{C}^\infty(\Omega)$ we have*

$$(K_\sigma \operatorname{curl} z, K_\sigma \operatorname{curl} z)_\Omega = 2\,(K_\sigma \operatorname{curl} z, \nabla\tau_\sigma \wedge z)_\Omega. \tag{5.13}$$

Proof. We have

$$\begin{aligned}(K_\sigma \operatorname{curl} z, K_\sigma \operatorname{curl} z)_\Omega &= \int_0^T ds\,((X_\sigma^s - Y_\sigma^s)\operatorname{curl} z, K_\sigma \operatorname{curl} z)_\Omega \\ &= \int_0^T ds \int_0^T d\xi\,((X_\sigma^s - Y_\sigma^s)\operatorname{curl} z, (X_\sigma^\xi - Y_\sigma^\xi)\operatorname{curl} z)_\Omega \\ &= 2\int_0^T ds \int_0^s d\xi\,((X_\sigma^s - Y_\sigma^s)\operatorname{curl} z, (X_\sigma^\xi - Y_\sigma^\xi)\operatorname{curl} z)_\Omega \\ &= 2\int_0^T ds\,((X_\sigma^s - Y_\sigma^s)\operatorname{curl} z, K_\sigma^s \operatorname{curl} z)_{\Omega^s[\sigma]}.\end{aligned} \tag{5.14}$$

As is clear, the field $\beta := (X_\sigma^s - Y_\sigma^s)\operatorname{curl} z$ is orthogonal to $\mathcal{C}\langle\Omega^s[\sigma]\rangle$. Moreover, it is smooth in $\Omega^s[\sigma]$, since it is solenoidal and satisfies (5.12). So we can apply Lemma 2 to the integrand:

$$((X_\sigma^s - Y_\sigma^s)\operatorname{curl} z, K_\sigma^s \operatorname{curl} z)_{\Omega^s[\sigma]} = ((X_\sigma^s - Y_\sigma^s)\operatorname{curl} z, \nabla\tau_\sigma \wedge z)_{\Omega^s[\sigma]}.$$

Substituting this in (5.14), we obtain

$$\begin{aligned}(K_\sigma \operatorname{curl} z, K_\sigma \operatorname{curl} z)_\Omega &= 2\int_0^T ds\,((X_\sigma^s - Y_\sigma^s)\operatorname{curl} z, \nabla\tau_\sigma \wedge z)_{\Omega^s[\sigma]} \\ &= 2\,(K_\sigma \operatorname{curl} z, \nabla\tau_\sigma \wedge z)_\Omega. \quad \square\end{aligned}$$

Applying (5.13) to $z \in \vec{C}^\infty(\Omega)$, we get

$$\|K_\sigma \operatorname{curl} z\|_\Omega^2 = 2\,(K_\sigma \operatorname{curl} z, \nabla\tau_\sigma \wedge z)_\Omega \leqslant C\,\|K_\sigma \operatorname{curl} z\|_\Omega \cdot \|z\|_\Omega.$$

Therefore,

$$\|K_\sigma \operatorname{curl} z\|_\Omega \leqslant C\,\|z\|_\Omega. \tag{5.15}$$

Lemma 4. *For any field $u \in \mathcal{C}$ the relations*

$$\|\operatorname{curl}(K_\sigma u)\|_\Omega \leqslant C\,\|u\|_\Omega \tag{5.16}$$

and

$$\nu \wedge (K_\sigma u)|_\Gamma = 0 \tag{5.17}$$

are valid.

Proof. Let $z \in \vec{C}^\infty(\Omega)$. By the symmetry (5.8) and the estimate (5.15) we have

$$\begin{aligned}|(K_\sigma u, \operatorname{curl} z)_\Omega| &= |(u, K_\sigma \operatorname{curl} z)_\Omega| \leqslant \|u\|_\Omega \cdot \|K_\sigma \operatorname{curl} z\|_\Omega \\ &\leqslant C\|u\|_\Omega \cdot \|z\|_\Omega.\end{aligned} \tag{5.18}$$

By the Riesz theorem, there exists $w \in \vec{\mathcal{H}}$ such that

$$(K_\sigma u, \operatorname{curl} z)_\Omega = (w, z) \qquad \forall z \in \vec{C}^\infty(\Omega). \tag{5.19}$$

This means that (in a generalized sense)

$$\operatorname{curl}(K_\sigma u) = w.$$

Moreover, inequality (5.18) implies

$$\|\operatorname{curl}(K_\sigma u)\|_\Omega = \|w\|_\Omega \leq C\|u\|_\Omega.$$

Now relation (5.19) rewritten as

$$(K_\sigma u, \operatorname{curl} z)_\Omega = (\operatorname{curl}(K_\sigma u), z) \quad \forall z \in \vec{C}^\infty(\Omega)$$

implies (5.17) (see the definition (5.2) and remark after it). □

Lemma 5. *Let $\sigma \subset \Gamma$. For any field $u \in \mathcal{C}$, we have*

$$\|\operatorname{div}(K_\sigma u)\|_\Omega \leqslant C\,\|u\|_\Omega. \tag{5.20}$$

Proof. By the definition of K_σ, for large enough T we have

$$K_\sigma u = \int_0^T X_\sigma^s\, ds\, u - \int_0^T Y_\sigma^s\, ds\, u.$$

The second term belongs to $\mathcal{C}$ and thus has vanishing divergence in Ω. By (5.9) the first term is equal to $(T - \tau_\sigma)\, u$. Then by formula (2.8) we have

$$\operatorname{div}(K_\sigma u) = \operatorname{div}((T - \tau_\sigma)\, u) = -\nabla\tau_\sigma \wedge u.$$

This completes the proof. □

Proof of Theorem 1. Suppose $u \in \mathcal{C}$. It follows from the estimates (5.16), (5.20) and boundary condition (5.17) that

$$\|K_\sigma u\|_F \leqslant \widetilde{C}\,\|u\|_\Omega.$$

Then, by the compactness of the embedding $F \subset \vec{\mathcal{H}}$ (Theorem 4), we conclude that $K_\sigma \in \mathfrak{K}(\mathcal{C}; \vec{\mathcal{H}})$. In view of (5.7) this completes the proof. □

5.2. Homomorphism $\dot{\pi}$

The proof of Theorem 3 uses a map $\dot{\pi}$, which is introduced here.

Let Y be the projection in $\vec{\mathcal{H}}$ onto $\mathcal{C}$. With a function $f \in C(\Omega)$ we associate an operator $Y[f] \in \mathfrak{B}(\mathcal{C})$ acting by the rule

$$Y[f]\, y := Y(fy), \quad y \in \mathcal{C}.$$

Define a map $\dot{\pi} : C(\Omega) \to \mathfrak{B}(\mathcal{C})/\mathfrak{K}$,

$$\dot{\pi}(f) := \pi(Y[f])$$

(recall that $\pi : \mathfrak{B}(\mathcal{C}) \to \mathfrak{B}(\mathcal{C})/\mathfrak{K}$ is the canonical projection).

For $f \in C(\Omega)$ we denote by $\check{f}$ the operator in $\vec{\mathcal{H}}$ that multiplies fields by f.

Lemma 6. (i) *For any $f \in C(\Omega)$ we have*

$$\check{f} - Y[f] \in \mathfrak{K}(\mathcal{C}; \vec{\mathcal{H}}). \tag{5.21}$$

(ii) *The mapping $\dot{\pi}$ is an isometric isomorphism on its image.*

Proof. (i) First we prove (5.21) for $f \in C^\infty(\Omega)$. Choose a finite open cover $\{U_j\}$ of the support of f such that every set of this cover is C^∞-diffeomorphic to a ball in $\mathbb{R}^3$ in the case, where $U_j \cap \Gamma = \emptyset$ or to a semiball $\{x \in \mathbb{R}^3 : |x| < 1,\ x^3 \geqslant 0\}$ otherwise. Choose a partition of unity $\zeta_j \in C_0^\infty(U_j)$ such that

$$0 \leqslant \zeta_j \leqslant 1, \qquad \sum_j \zeta_j \Big|_{\operatorname{supp} f} = 1.$$

It is clear that

$$\check{f} - Y[f] = \sum_j (\check{\zeta_j f} - Y[\zeta_j f]),$$

and the functions $\zeta_j f$ belong to $C_0^\infty(U_j)$. Thus, it is necessary to prove (5.21) for a function f supported in an open set U C^∞-diffeomorphic to a ball or a semiball. In this case, for any $y \in \mathcal{C}$ we have

$$(fy - Y[f]\, y)|_U = \nabla p_y, \quad p_y \in H^1(U). \tag{5.22}$$

Indeed, by the decomposition (2.11) we have

$$fy - Y[f]\, y = \nabla q_y + h_y, \quad q_y \in H_0^1(\Omega),\ h_y \in \mathcal{D}.$$

Being harmonic, the field h is smooth in Ω and satisfies $\operatorname{curl} h = 0$ (see comments after (2.11)). Since U is contractible, by the Poincare lemma we have

$$h_y|_U = \nabla\tilde{q}_y|_U.$$

Together with the previous equation, this yields (5.22) with $p_y = q_y + \tilde{q}_y$. Now suppose that the set U intersects Γ. Since h_y satisfies the Dirichlet boundary condition we have $\nu \wedge h_y|_\Gamma = 0$ and thus

$$\nu \wedge \nabla\tilde{q}_y|_{U\cap\Gamma} = 0;$$

hence, q_y is constant on $U \cap \Gamma$, which leads to the relation

$$p_y|_{U\cap\Gamma} = (q_y + \tilde{q}_y)|_{U\cap\Gamma} = \tilde{q}_y|_{U\cap\Gamma} = \text{const}$$

(recall that $q_y \in H_0^1(\Omega)$).

The function p_y in (5.22) is uniquely determined up to additive constant, which can be chosen so that

$$p_y|_{U\cap\Gamma} = 0 \tag{5.23}$$

if $U \cap \Gamma \neq \emptyset$, and

$$\int_U p_y \, dx = 0$$

otherwise. The Friedrichs and Poincaré inequalities imply that, in both cases, there is a constant C such that

$$\|p_y\|_U \leqslant C\|\nabla p_y\|_U = C\|fy - Y[f]\,y\|_U \leqslant C\|\check{f} - Y[f]\| \cdot \|y\|_\Omega.$$

Therefore, the mapping

$$y \mapsto p_y \tag{5.24}$$

is continuous from $\mathcal{C}$ to $H^1(U)$.

Now assume that a sequence y_n weakly converges to zero in $\mathcal{C}$. By the continuity of the map (5.24) the sequence p_{y_n} weakly converges to zero in $H^1(U)$, and, by the compactness of the embedding $H^1(U) \subset L_2(U)$, this implies

$$\|p_{y_n}\|_U \to 0, \quad n \to \infty. \tag{5.25}$$

Next, we have

$$\|fy_n - Y[f]\,y_n\|_\Omega^2 = (fy_n, fy_n - Y[f]\,y_n)_\Omega = (fy_n, \nabla p_{y_n})_U.$$

In the last relation we used (5.22) and the inclusion $\operatorname{supp} f \subset U$. Integrating by parts in this inner product, and applying formula (2.8) and the relation $\operatorname{div} y_n = 0$, we arrive at

$$(fy_n, \nabla p_{y_n})_U = -\int_U \nabla f \cdot y_n\, p_{y_n}\, dx \leqslant M\|y_n\|_\Omega \cdot \|p_{y_n}\|_U,$$

where M depends only on f (note that the estimated product is nonnegative owing to the previous calculation). The integral over ∂U vanishes since f vanishes on $\partial U \setminus \Gamma$ and in the case $U \cap \Gamma \neq \emptyset$ we have (5.23). The right hand-side of the latter inequality tends to zero, because the norms of y_n are bounded and (5.25) takes place. Then, with regard to the result of the previous calculation, we get the relation

$$\|fy_n - Y[f]\,y_n\|_\Omega \to 0, \quad n \to \infty,$$

which shows that the operator $\check{f} - Y[f]$ is compact.

Now let us consider the case $f \in C(\Omega)$ in (5.21). The function f can be approximated in $C(\Omega)$ by functions $f_n \in C^\infty(\Omega)$. The operators of multiplication by f_n tend to the operator of multiplication by f in the operator norm. Hence, the operator $\check{f} - Y[f]$ is compact as a limit of compact operators.

(ii) Now turn to the second assertion. Here we prove the following properties:

$$\dot{\pi}(\alpha f + \beta g) = \alpha\dot{\pi}(f) + \beta\dot{\pi}(g),$$
$$\dot{\pi}(fg) = \dot{\pi}(f)\,\dot{\pi}(g),$$
$$\|\dot{\pi}(f)\| = \|f\|,$$

where $f, g \in C(\Omega)$, $\alpha, \beta \in \mathbb{R}$. The first and second relations follow from (5.21). For example, consider the second one. We show that

$$Y[f]\,Y[g] - Y[fg] \in \mathfrak{K}. \tag{5.26}$$

By (5.21), we have

$$Y[f]\,Y[g]=(f+K_1)\,Y[g]=fY[g]+K=f(g+K_2)+K=fg+\widetilde{K},$$

where $K_1, K_2, K, \widetilde{K}\in\mathfrak{K}(\mathcal{C},\vec{\mathcal{H}})$. Applying (5.21) to the function fg, we obtain (5.26).

Consider the third property. We can restrict ourselves to smooth f since the mapping $\dot{\pi}$ is bounded. The latter follows from the obvious inequality

$$\|\dot{\pi}(f)\|\leqslant\|f\|.$$

Let us establish the opposite inequality. We need to show that for any compact operator $K\in\mathfrak{K}$ we have

$$\|Y[f]+K\|\geqslant\|f\|. \tag{5.27}$$

If f is constant, (5.27) is trivial. Otherwise consider an arbitrary point $x_0\in\Omega\setminus\Gamma$ such that $\nabla f(x_0)\neq 0$. Choose a sequence of functions $\varphi_j\in C_0^\infty(\Omega\setminus\Gamma)$ such that supp φ_j shrink to x_0 as $j\to\infty$. Introduce the fields

$$y_j:=\nabla f\wedge\nabla\varphi_j.$$

Functions φ_j can be chosen such a way that every field y_j does not vanish identically. Owing to (2.9), we have $\operatorname{div} y_j=0$. Since the supp y_j tend to x_0 as $j\to\infty$, for sufficiently large j the fields y_j belong to $\mathcal{C}$. Further, we have

$$f\,y_j=f\nabla f\wedge\nabla\varphi_j=\frac{1}{2}\nabla(f^2)\wedge\nabla\varphi_j,$$

so by (2.9) $\operatorname{div}(fy_j)=0$ and for large j the fields fy_j also belong to $\mathcal{C}$. Hence

$$Y[f]y_j=Y(fy_j)=fy_j. \tag{5.28}$$

Consider a normed sequence

$$\tilde{y}_j=y_j/\|y_j\|.$$

Obviously, the sequence $\tilde{y}_j$ weakly converges to zero in $\mathcal{C}$. Therefore $K\tilde{y}_j\to 0$ in $\mathcal{C}$. With regard to (5.28), this yields

$$\|(Y[f]+K)\,\tilde{y}_j\|=\|f\tilde{y}_j+K\tilde{y}_j\|\to|f(x_0)|,\quad j\to\infty.$$

Since $\|\tilde{y}_j\|=1$, we arrive at the inequality $\|Y[f]+K\|\geqslant|f(x_0)|$. This occurs for all points x_0, at which f has a nonzero gradient. Since f is nonconstant and Ω is connected for any $\delta>0$, there is x_0 such that

$$\nabla f(x_0)\neq 0,\qquad |f(x_0)|>\|f\|-\delta.$$

Turning $\delta\to 0$, we obtain (5.27). □

5.3. *Proof of Theorem 3*

To prove Theorem 3 it suffices to show that the map $\dot{\pi}$ is an extension of the map $\tau_\sigma\mapsto\pi\varepsilon_\sigma$. To this end, let us show that $\varepsilon_\sigma-Y[\tau_\sigma]\in\mathfrak{K}$. Indeed, we have

$$\varepsilon_\sigma-Y[\tau_\sigma]=\varepsilon_\sigma-\check{\tau}_\sigma+\check{\tau}_\sigma-Y[\tau_\sigma]$$

and, by Theorem 1 and Lemma 6(i), there is a sum of two compact operators from $\mathfrak{K}(\mathcal{C};\vec{H})$ on the right-hand side. Now Theorem 3 follows from Lemma 6(ii) and the fact that the algebra $\dot{\mathfrak{E}}$ is generated by the elements $\pi\varepsilon_\sigma$.

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